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More on the Fundamental Theorem of Calculus

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In a note [**1**] in the MONTHLY, Botsko and Gosser point out that the standard version of the Fundamental Theorem of Calculus holds when the usual derivative is replaced by the right-hand derivative. We would like to point out that by making a *slight* alteration in the usual definition of the Riemann integral, we can obtain an integral for which the Fundamental Theorem of Calculus holds in *full generality*.

We begin by recalling one of the common definitions of the Riemann integral. If $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ is a partition of $[a, b]$, the *mesh* of P is $\max\{x_i - x_{i-1} : i = 1, \ldots, n\}$.

DEFINITION 1. A function $f : [a, b] \to \mathbb{R}$ is *Riemann integrable* over $[a, b]$ if there exists $A \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if P is a partition of mesh less than δ and if $t_i \in [x_{i-1}, x_i]$, then

$$\left| \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - A \right| < \varepsilon.$$

The number A is called the *Riemann integral* of f and is denoted by $\int_a^b f$.

In order for a function $f : [a, b] \to \mathbb{R}$ to be Riemann integrable it is necessary that whenever the interval $[a, b]$ is partitioned into subintervals of length less than δ, the Riemann sums $\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$ approximate the integral of f within ε. It is this requirement of being able to uniformly partition the interval that limits the scope of the Riemann integral. It would be much more desirable to somehow allow "variable length" partitions; for example, if one were attempting to approximate the area under the graph of $f(x) = 1/\sqrt{x}$, $0 < x \le 1$, it would be natural to take the subintervals in an approximating partition to be very fine near the singularity $x = 0$. By making a slight modification in the definition above, we can easily achieve the ability to employ such variable length partitions.

First, note that the requirement in Definition 1 that the partition P have mesh less than δ can be replaced by the condition:

$$[x_{i-1}, x_i] \subseteq \left(t_i - \frac{\delta}{2}, t_i + \frac{\delta}{2} \right) \quad \text{where} \quad t_i \in [x_{i-1}, x_i]. \tag{1}$$

Now we can achieve the desired variable length partition by merely replacing the constant δ in (1) by a positive-valued function $\delta : [a, b] \to \mathbb{R}$, i.e., we replace (1) by:

$$[x_{i-1}, x_i] \subseteq (t_i - \delta(t_i), t_i + \delta(t_i)) \quad \text{where} \quad t_i \in [x_{i-1}, x_i]. \tag{1'}$$

By varying $\delta(t)$ as t varies along the interval $[a, b]$, the lengths of the subintervals in the partition satisfying (1′) will vary with $\delta(t_i)$.

We give the formal definition of the resulting integral. A tagged partition of $[a, b]$ is a finite set $T = \{x_0, x_1, \ldots, x_n; t_1, t_2, \ldots, t_n\}$ such that $\{x_0, x_1, \ldots, x_n\}$ is a partition of $[a, b]$ and $t_i \in [x_{i-1}, x_i]$; the point t_i is said to be a *tag* for the subinterval $[x_{i-1}, x_i]$. Any positive-valued function $\delta : [a, b] \to \mathbb{R}$ is called a *gauge* on $[a, b]$. We say that a tagged division T is *δ-fine* if (1′) is satisfied.

DEFINITION 2. A function $f : [a, b] \to \mathbb{R}$ is *gauge-integrable* over $[a, b]$ if there exists $A \in R$ such that for every $\varepsilon > 0$ there exists a gauge δ on $[a, b]$ such that if T is a δ-fine tagged partition of $[a, b]$, then

$$\left| \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - A \right| < \varepsilon.$$

The number A is called the (gauge) integral of f and is denoted by $\int_a^b f$. The gauge integral is also referred to as the generalized Riemann integral **[9]** or the Riemann complete integral **[5]**. From Definition 1, we see that a function is Riemann integrable iff it is gauge integrable with respect to constant-valued gauges. Note also that if a function is gauge integrable then its integral is the limit of a sequence of Riemann sums. One technicality must be taken care of in order for Definition 2 to make sense: it must be shown that every gauge δ has at least one δ-fine tagged partition. Currently this observation is ascribed to Pierre Cousin **[2]** but it may go somewhat earlier. The lemma has a curious habit of rediscovery: for example, each of the articles **[3]**, **[7]**, **[12]**, **[14]** contains a fresh account with similar applications. The proof requires only a compactness argument (based on the Bolzano-Weierstrass or Heine-Borel theorems) and indeed the lemma is equivalent to these theorems. The reader can find an elementary proof in **[9]**.

Before proceeding to the Fundamental Theorem of Calculus, consider the integrability of the Dirichlet function: $f(x) = 1$ for $0 \le x \le 1$ and rational and $f(x) = 0$ for $0 \le x \le 1$ and irrational. This is the most common example given of a bounded function that is not Riemann integrable, and will, therefore, furnish a comparison of the gauge and Riemann integrals. Let $\varepsilon > 0$ be given and for x irrational, set $\delta(x) = 1$. Let $\{z_i\}_{i=1}^{\infty}$ be an enumeration of the rationals in $[0,1]$ and set $\delta(z_i) = \varepsilon/2^{i+1}$. Now suppose that $T = \{x_0, x_1, \ldots, x_n; t_1, t_2, \ldots, t_n\}$ is a δ-fine tagged division of $[0,1]$. If t_i is not rational, the term $f(t_i)(x_i - x_{i-1})$ in the Riemann sum of f with respect to T is 0; if t_i is rational and $t_i = z_j$, then the term $f(t_i)(x_i - x_{i-1})$ in the Riemann sum is less than $2\delta(z_j) = \varepsilon/2^{j+1}$. Thus, we have

$$\left| \sum_{i=0}^{n-1} f(t_i)(x_i - x_{i-1}) \right| < 2 \sum_{j=1}^{\infty} \varepsilon/2^{j+1} = \varepsilon,$$

where the factor 2 is necessary since each z_j may be the tag for two of the subintervals in the partition. This implies that f is gauge integrable with $\int_0^1 f = 0$. Note that the gauge δ is definitely not a constant-valued gauge.

We now show that the Fundamental Theorem of Calculus is valid for the gauge integral in full generality. For this we require the following lemma which plays the role of the Mean Value Theorem in the usual proofs of the Fundamental Theorem of Calculus for the Riemann integral.

LEMMA 3 (STRADDLE LEMMA). *Let $F:[a,b]\to\mathbb{R}$ be differentiable at $z\in[a,b]$. Then for each $\varepsilon>0$, there is a $\delta>0$ such that*

$$|F(v)-F(u)-F'(z)(v-u)|\le\varepsilon(v-u),$$

whenever $u\le z\le v$ and $[u,v]\subseteq[a,b]\cap(z-\delta,z+\delta)$.

Proof. Since F is differentiable at z, there is a $\delta>0$ such that

$$|(F(x)-F(z))/(x-z)-F'(z)|<\varepsilon$$

for $0<|x-z|<\delta$, $x\in[a,b]$. If $z=u$ or $z=v$, the conclusion is immediate so suppose $u<z<v$. Then,

$$\begin{aligned}|F(v)&-F(u)-F'(z)(v-u)|\\&\le|F(v)-F(z)-F'(z)(v-z)|+|F(z)-F(u)-F'(z)(z-u)|\\&<\varepsilon(v-z)+\varepsilon(z-u)=\varepsilon(v-u).\end{aligned}$$

The geometric interpretation of the Straddle Lemma is clear. If the points u and v "straddle" z, then the slope of the chord between the points $(u,f(u))$ and $(v,f(v))$ is close to the slope of the tangent line at $(z,f(z))$.

THEOREM 4 (FUNDAMENTAL THEOREM OF CALCULUS). *If $F:[a,b]\to\mathbb{R}$ is differentiable on $[a,b]$, then F' is gauge integrable over $[a,b]$ and $\int_a^b F'=F(b)-F(a)$.*

Proof. Let $\varepsilon>0$. For $z\in[a,b]$, let $\delta(z)>0$ be the δ given by the Straddle Lemma. Suppose $T=\{x_0,x_1,\dots,x_n;\ t_1,t_2,\dots,t_n\}$ is a δ-fine tagged partition of $[a,b]$. Then by the Straddle Lemma,

$$\begin{aligned}&\left|\sum_{i=1}^n F'(t_i)(x_i-x_{i-1})-(F(b)-F(a))\right|\\&\quad=\left|\sum_{i=1}^n\{F'(t_i)(x_i-x_{i-1})-(F(x_i)-F(x_{i-1}))\}\right|\\&\quad<\sum_{i=1}^n\varepsilon(x_i-x_{i-1})=\varepsilon(b-a),\end{aligned}$$

and the conclusion follows.

Note that in general the gauge δ constructed above is not a constant-valued gauge and depends on the differentiability properties of F.

The versions of the Fundamental Theorem of Calculus for both the Riemann and Lebesgue integrals require the hypothesis that the derivative F' is integrable; it is part of the conclusion of Theorem 4 that the derivative F' is gauge integrable. For example, the derivative of the function $F(t)=t^2\cos(\pi/t^2)$, $0<t\le1$, $F(0)=0$ is gauge integrable, but is not integrable for either the Riemann or Lebesgue integrals.

To further illustrate the utility of the gauge integral, we now proceed to generalize Theorem 4 by allowing the function F to be nondifferentiable at a countable number of points.

THEOREM 5. *Let $F:[a,b]\to\mathbb{R}$ be differentiable except perhaps at countably many points of $[a,b]$. Let $G:[a,b]\to\mathbb{R}$ be such that $G(x)=F'(x)$ when F is differentia-*

ble at x. If F is continuous, then G is gauge integrable over $[a, b]$ *with* $\int_a^b G = F(b) - F(a)$.

Proof. The proof is not substantially different and requires only some arithmetic to take care of the exceptional set $N = \{z_i : i = 1, 2, \dots\}$ where F may fail to be differentiable. Let $\varepsilon > 0$. For $x \notin N$ define $\delta(x) > 0$ by the Straddle Lemma as before. For $z_i \in N$, choose $\delta(z_i) > 0$ such that $|G(z_i)| 2\delta(z_i) < \varepsilon/2^{i+2}$ and

$$|F(z_i) - F(z_i + h)| < \varepsilon/2^{i+3}$$

for $|h| \le \delta(z_i)$ (by continuity). Suppose that $T = \{x_0, x_1, \dots, x_n; t_1, t_2, \dots, t_n\}$ is a δ-fine tagged partition of $[a, b]$. Then as before,

$$\left|\sum_{i=1}^{n} G(t_i)(x_i - x_{i-1}) - (F(b) - F(a)\right|$$
$$= \left|\sum_{i=1}^{n} \left(G(t_i)(x_i - x_{i-1}) - (F(x_i) - F(x_{i-1}))\right)\right|. \tag{2}$$

We break the sum on the right-hand side of (2) into two parts. Let Σ' denote the sum of the terms with tags $t_i \notin N$, and let Σ'' denote the sum of the terms with tags $t_i \in N$. As before the sum Σ' is less than $\varepsilon(b - a)$. For Σ'', if $t_i = z_j$, then

$$|G(t_i)(x_i - x_{i-1})| \le |G(z_j)| 2\delta(z_j) < \varepsilon/2^{j+2}$$

and

$$|F(x_i) - F(x_{i-1})| \le |F(x_i) - F(z_j)| + |F(z_j) - F(x_{i-1})| < 2\varepsilon/2^{j+3}.$$

Hence,

$$\Sigma'' \le 2\left(\sum_{j=1}^{\infty} \varepsilon/2^{j+2} + \sum_{j=1}^{\infty} \varepsilon/2^{j+2}\right) = \varepsilon,$$

where the factor of 2 accounts for the fact that each z_j may be the tag for at most two of the subintervals. Thus, the sum in (2) is less than $\varepsilon(b - a) + \varepsilon$ and the result follows.

Both versions of the Fundamental Theorem of Calculus given in Theorems 4 and 5 are well known for the gauge integral and can be found in **[9]**. There is also a divergence version of the Fundamental Theorem of Calculus for a gauge-type integral in n dimensions given in **[13]**.

The functions $F(x) = 2x^{1/2}$, $0 \le x \le 1$, and $G(x) = x^{-1/2}$ for $0 < x \le 1$ and $G(0) = 0$ provide a simple example where Theorem 5 is applicable but the more familiar version of the Fundamental Theorem of Calculus is not. Note that in this case, the integral

$$\int_0^1 x^{-1/2}\,dx = \int_0^1 G = \int_0^1 F' = 2$$

is computed without resorting to the limiting technique required by the Riemann approach.

Note that the continuity assumption in Theorem 5 is important. For example, if $F_1(x) = x + 1$ for $0 \le x \le 1$ and $F_1(x) = -x$ for $-1 \le x < 0$, then $F_1'(x) = G(x)$ except for $x = 0$, but $\int_{-1}^1 G = 0$ while $F_1(1) - F_1(-1) = 1$.

As can be seen from the definition, the gauge integral has very much the same flavor as the Riemann integral, being obtained from a slight modification of the Riemann integral, and does not require a lot of technical apparatus for its introduction as is the case for the Lebesgue integral. However, despite the elementary appearance of the gauge integral, it leads to a very powerful theory of integration which encompasses the Riemann integral, the Cauchy-Riemann (improper Riemann) integral, and the Lebesgue integral. For this reason, the gauge integral would seem to be a very reasonable candidate for inclusion in an introductory real analysis course; it is as conceptually easy to describe as the Riemann integral and yet possesses all of the powerful properties of the Lebesgue integral including the Monotone and Dominated Convergence Theorems.

Remarkably, this simple modification of the Riemann integral was not introduced until approximately a century after Riemann's introduction of his integral in 1854. The gauge integral was independently introduced by Kurzweil **[6]** and Henstock **[4]**; Kurzweil used the integral to treat some questions in ordinary differential equations but did not develop any of the deep properties of the integral; Henstock established the convergence theorems for the integral.

The interested reader can find very readable expositions of the gauge integral in **[5]**, **[8]**, **[9]**. E. J. McShane also treats a "gauge-like" integral in **[10]**, **[11]**; he alters the definition above by dropping the requirement that the tag t_i belongs to its corresponding subinterval. The resulting integral is, surprisingly enough, exactly equivalent to the classical Lebesgue integral.

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A Strong Inverse Function Theorem

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A linear transformation from $\mathbb{R}^n$ into the same space $\mathbb{R}^n$ is one-to-one if and only if its matrix relative to, say, the standard basis has nonzero determinant. In this