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Real Variable Contributions of G. C. Young and W. H. Young

Andrew M. Bruckner[1] and Brian S. Thomson[2]

[1]Department of Mathematics, University of California, Santa Barbara, California 93106, USA
[2]Department of Mathematics, Simon Fraser University, Burnaby, B.C., Canada, V5A 1S6

§1. **Introduction.** The Youngs[1] began to work on real functions in the earliest years of the 20th century. Then, as now, it was an unfashionable subject. Writing of this time Hardy [40, p. 224] (see also [1]) says "these subjects were not popular, even in France, with conservatively minded mathematicians; in England they were regarded almost as a morbid growth in mathematics ...." Fashions are dictated, of course, by vested interests, pride and ignorance. It is hard to imagine, from our perspective at the beginning of the 21st century, a more profitable time to study this field than at the beginning of the previous century; Cantor's set theory was very much in the air and all of the important basic tools were being provided by Baire, Borel and Lebesgue. The whole field of what was then called "the theory of functions of a real variable" was reworked and rewritten in those first decades. The Youngs played a major role in this effort.

It is beyond our ability to present a complete account of their influence on this field. Much of their work was "influential" in the sense that they popularized and made better known the seminal contributions of Cantor or the important work in integration theory that Lebesgue had produced or the category ideas of Baire. Many of their papers are extensions or applications of these themes with new proofs or new techniques. An indication of their impact is evident in Hobson [43] which for many years was the main English language reference work on real functions: there are 139 citations of their works in these two volumes. This necessary professional work does not often lead to long lasting fame and renown and by now the sources have blurred considerably. A modern graduate course in real functions doubtless owes much to their activity but it is only infrequently explicit.

[1]This essay on Youngs' influence on some aspects of real analysis was originally intended to accompany an edition of the collected works of the Youngs planned as a four volume work with essays covering various aspects of their contributions. Unfortunately this ambitious project had to be cancelled. In its place, a one volume edition of Selected Papers [1] was published by Presses Polytechniques et Universitaires Romandes in 2000, edited by S. D. Chatterji and H. Wefelscheid. Professor Chatterji has done a major service by summarizing in a short introduction many themes that run through the vast body of work that the Youngs produced in their careers.

E-mail addresses: [1]bruckner@math.ucsb.edu
[2]thomson@sfu.ca

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Our focus in this review shall be to present those moments in their work when a truly original conception has arisen which has then led to a deeper pursuit by later analysts. As far as possible we shall trace many of their ideas from their sources through to the large real analysis literature that the Youngs can be considered as having inspired or anticipated. The bibliography is extensive but should not be taken as complete.

The joint efforts of the Youngs in this research counts as one of the most fruitful and longest collaborations known. Perhaps the only parallel is with Hardy and Littlewood. One of the "axioms" that directed the Hardy–Littlewood collaboration was that for a paper to appear under their common name it was indifferent whether one of them had contributed the least bit to the contents. Accordingly it is impossible to separate out from that collaboration the individual efforts. A similar claim must be made for the Young collaboration. In this case the axiom that one should assume is that a paper with just his name or both of their names must be considered as joint work with no hope of ever sorting out the source of the ideas.

Their son, L. C. Young [76] explains it thus:

> This takes me to another story that I may be excused for regarding as romantic, the story of my parents. ...I dislike blowing the family trumpet: there are generous obituaries by G. H. Hardy and by Dame Mary Cartwright in the Journal of the London Mathematical Society and there is a fine recent biography by Grattan-Guiness in the Annals of Science (1972). My parents published, in addition to the *Theory of Sets of Points* (1906), two other mathematical books and 214 papers; 18 papers were my mothers and 13 were joint. This is partly because my mother wanted the credit to go to my father; it would be fairer to say that about a third of the material, and virtually all of the writing up of the final version, was due to my mother: this last was made necessary by the fact that my father also had to make a living. There was no such thing as a research grant. To give my father every possible chance, my mother took on the work of a small army of assistants and secretaries; occasionally he slipped in her name as co-author — she certainly did not.

Even with the few exceptional papers that Grace published on her own we must consider all their work as a collaboration and their reputation as analysts stays linked. For example one of the most famous theorems that is their legacy, the Denjoy–Young–Saks theorem, was work that appeared under her name; but the roots of the ideas are too entangled with papers published earlier to allow a separation even in this case.

Thus even if in this review we have occasion to refer to "his paper" or "her theorem" it might be more appropriate for the reader to assume that the work was joint.

§2. **Inner limiting sets.** According to Hobson [42] the notion of an inner limiting set was suggested to Young by an observation of Borel on Liouville numbers. For any $\lambda > 0$ the union of the open intervals

$$(p/q - \lambda/q^2, p/q + \lambda/q^2)$$

taken over all rationals p/q contains, in addition to the rationals themselves, the transcendental Liouville numbers. (For a modern account of this see [62, pp. 6–9].) This suggests a general study of the nature of sets formed as the limits of decreasing sequences of unions of open intervals. It is this that in [W 1903ii] is called an inner limiting set.

The main result of that paper is that such a set is countable unless it contains a subset dense in itself in which case it has the cardinality of the continuum. (In particular the continuum hypothesis holds for these sets.) Nowadays we recognize this assertion as a statement about $\mathcal{G}_\delta$ sets; the parallel notion of an $\mathcal{F}_\sigma$ was called by Young an outer limiting set. The introduction of these ideas in analysis is generally attributed to him by the authorities of the time. Kuratowksi [46] states this explicitly; Hausdorff [41, p. 156] pays credit by referring to absolute $\mathcal{G}_\delta$ sets as "Young sets" and both he and Saks [68, p. 55] attribute the cardinality observation to Young.

The first application of these notions in real variable theory appears in [W 1903iii] prepared only shortly after [W 1903ii] . Improving and clarifying a result of Brodén he shows that the set of points where one of the derivates of a continuous function is infinite is an inner limiting set. It is apparently one of the first applications of the set classifications to a problem in analysis. Later in [W 1903iv] he shows that given any inner limiting set E (i.e., a set of type $\mathcal{G}_\delta$) there is a function continuous at every point of E and discontinuous everywhere else.

The classification of sets continues in [W 1905i] but receives its proper expression in [W 1913xi] and [W 1916iv] where he essentially discusses the classification of Borel sets in his language and from his point of view; the parallel classification of functions in terms of monotone limits also is developed here. He labels inner and outer limiting sets as i and o sets, labels inner and outer limits of sequences of o and i sets as io and oi sets and so on; similarly upper and lower semicontinuous functions are labeled as u and l functions and monotone limits of these functions as ul and lu functions. In the appendix to the last paper the relation with the Borel and Baire classifications is expanded on. Evidently it was only after the submission of this paper that he was made aware of the major contribution of Lebesgue [48] in this area.

These tools, the classification of sets and functions, have in later years been applied extensively in the study of derivatives. Young (along with Denjoy on the continent) can be considered as the originator of much of this. Certainly the result in [W 1903iii] is one of the first. Later observations in [W 1913xi, p. 276] concern the sets of points $\{x : f^+(x) \geq k\}$ and $\{x : f^+(x) = k\}$ for the Dini derivate f^+ of a continuous function f, described there as an i set an io set respectively. The equivalent observation that the derivate f^+ is a ul function follows from [W 1913xi, p. 279].

We might mention, too, in this discussion of the inner limiting sets that the Youngs came close to characterizing the notion of a scattered set of real numbers. A set is scattered (in Cantor's original language *separierte Mengen*) if it contains no nonempty subset dense-in-itself. The Youngs made extensive use of such notions, including one-sided versions, in their study of real functions; see [W 1908xiii] and [G 1914ii] for example. In their book on set theory [77, p. 65] they state that a countable set is scattered if and only if it is an inner limiting set; Brouwer pointed out to Mrs. Young that the proof is incomplete (see [77, p. 298]). The correct theorem, which they did not quite realize, is that a set of real numbers is scattered if and only if it is a countable, inner limiting set. Later contributions of Denjoy and Sierpiński clarify the nature of these sets.

§3. **Associated sets.** After Young most such results as those just mentioned, classifying derivates and sets arising in their study, use Baire classes and Borel classes. There is an extensive literature now that can be considered as started from these preliminary observations.

Generalizing the results from [W 1913xi], Sierpinski [69] shows that if f is a function in Baire class α then the Dini derivative f^+ is in Baire class $\alpha+3$, so that in particular the sets $\{x : f^+(x) \geq k\}$ are in Borel class $\alpha+3$. Banach [6] improved this, for bounded functions, to class $\alpha+2$; Mišik [56] extended this by removing the boundedness assumption and by showing that, in fact, f^+ is upper semi-Borel of class $\alpha+1$. Extensions of this to generalized derivatives are given in Alikhani-Koopaei [4].

As we pass from the structure of the Dini derivates to other generalized derivates a variety of different situations can occur. Hajek [38] shows that for a completely arbitrary function f the upper bilateral derivates are in Baire class 2 (Banach had previously noted that they are measurable). Generalized versions of this too have appeared (eg. [5] and [49]). For symmetric derivates the situation differs again: nonmeasurable functions can have nonmeasurable symmetric derivates, measurable functions have measurable symmetric derivates but a Baire 2 function exists with symmetric derivates that are not even Borel measurable (Laczkovich [47]). Classification of the ordinary and approximate partial derivates of functions of several variables can be found in [68, pp. 300–313], [65], and [57]. Here one finds, surprisingly, that the approximate partials of a function f much more faithfully reflect the class of f than do the ordinary partials.

The deepest and most important works dealing with the associated sets arising in differentiation theory have been provided by Zahorski [78] and Preiss [63]. Zahorski obtained complete characterizations of the sets of the form

$$\{x : F'(x) < k\} \quad \text{and} \quad \{x : F'(x) > k\}$$

for functions F with bounded derivatives as well as some necessary conditions for other classes of derivatives. Some of Zahorski's principal techniques were actually anticipated by Choquet [18] who obtained a number of deep properties of derivatives. Preiss completed this work obtaining, among other things, a complete description of the nature of the sets $G = \{x : F'(x) = +\infty\}$, $E = \{x : F'(x) > k\}$ and the set S of points x where $\lim_{h\to 0+} F(x+h) > F(x)$ or $\lim_{h\to 0+} F(x-h) < F(x)$ for a function F that has a finite or infinite derivative at each point. (Preiss showed, too, that the same descriptions for G, E and S applies to the corresponding sets for the approximate derivative.) Note that this result brings us back to the Youngs themselves since they observed long ago that such a set S must be denumerable and that G would have to be a measure zero $\mathcal{G}_\delta$ for an arbitrary function F.

§4. **Characterizations of derivatives.** The article [W 1911vii] opens with a clear statement of a research program that has intrigued several generations. The problem is so well articulated by Young himself that he has been quoted extensively already in two accounts of this subject (Fleissner [27] and [10, Chapter 7]) and we can hardly resist again. The quote is from the opening passages of [W 1911vii, pp. 360–361] (footnotes ours).

> Recent research [of Lebesgue and Vitali] has provided us with a set of necessary and sufficient conditions that a function may be the indefinite integral... of another function and the way has thus been opened to important developments. The corresponding, much more difficult, problem of determining necessary and sufficient conditions that a function may be a differential coefficient, has barely been mooted; indeed, though we know a number of necessary conditions no set even of sufficient conditions has to my knowledge ever been formulated, except that involved in the obvious statement that a continuous function is a differential coefficient . The necessary conditions in question are of considerable importance and interest. A function which is a differential coefficient has, in fact, various striking properties. It must be pointwise discontinuous with respect to every perfect set[2]; it can have no discontinuities of the first kind; it assumes in every interval all values between its upper and lower bounds in that interval[3]; its value at every point is one of the limits on both sides of the values in the neighbourhood; its upper and lower bounds, when finite, are unaltered, if we omit the values at any countable set of points; the points at which it is infinite form an inner limiting set of content zero[4]. From these necessary conditions we are able to deduce much valuable information as to when a function is certainly not a differential coefficient. ... These conditions do not, however, render us any material assistance, even in answering the simple question as to whether the product of two differential coefficients is a differential coefficient, and this not even in the special case in which one of the differential coefficients is a continuous function.

The rest of the article following this quote is devoted to the problem raised in the last sentence here and which we shall explore in the next section. But this passage rings out clearly as a call for a program of research that has inspired many authors. Much of what we now discuss can be found in the article [11] which deals entirely with the problem and its ramifications, illustrating how much has happened in that area because of the Young problem. Here we have, as is often the case, an instance in which a single problem Young identified has influenced future research.

What exactly is a derivative? The catalogue of necessary conditions that was known to the Youngs has now been enlarged significantly; there are scarcely any new sufficient conditions. Perhaps the only sufficient condition that they might not have then known

[2] i.e., it must be Baire 1.

[3] i.e., it has the Darboux property.

[4] i.e., a $\mathcal{G}_\delta$ set that has measure zero (cf. [10, p. 229].)

is that a bounded approximately continuous function is a derivative (Denjoy [23]). The general problem however remains unsolved.

Young may have been then the first to realize the importance of the problem of characterizing derivatives. He had already noted in earlier papers that a number of existing theorems stated for continuous functions needed only the hypothesis that some function was a derivative. One finds in [W 1911viii], a study of functions defined by integrals, a number of theorems stated under the assumption that a function central to the statement of the theorem "be a differential coefficient" (i.e., a derivative). As he puts it

> It will be noticed ... we have adopted as one of our assumptions the condition that the integrand should be a differential coefficient with respect to one of the variables. ... In the present state of our knowledge we cannot give any but very special sets of sufficient conditions that a function should have the property of being a differential coefficient, so that the introduction of a condition of this form is not often of direct use in practice. Its importance in theory is, however, not affected by these considerations, and it has on other grounds seemed to me desirable, that, when we are concerned with a neighbourhood, it is the fact of a function being a differential coefficient, and not its continuity, that we usually require.

Later in his presidential address [W 1926iii] he again refers to the characterization problem.

One would like to obtain intrinsic properties of a function that can tell us whether or not a function is a derivative. To date no useful characterization has been found although much has been written on the subject. What causes this problem to be so difficult?

Most important classes of functions admit numerous characterizations. Continuous functions can be described in terms of associated sets or approximation by polynomials. Measurable functions too can be defined in terms of associated sets, or via Luzin's theorem, or a.e. approximate continuity, or as a.e. derivatives. Baire 1 functions have several well known characterizations (associated sets, limits of continuous functions, relative continuity points in each perfect set). Many more examples come easily to mind.

One difference that is a source of difficulty is contained in the fact that the class of derivatives is not closed under many familiar operations: it is not closed under multiplication, nor composition (inside or outside) with homeomorphisms. Banach in 1921 had shown that for a bounded derivative f the square f^2 is a derivative if and only if f is approximately continuous; this supplies both observations for multiplication and composition. Actually most derivatives f have the property that for every strictly convex h the function $h \circ f$ fails to be a derivative on any interval (see [10, p. 144]).

The most important attack on the problem to date was made by Zahorski [78]. He attempted to characterize the class of finite derivatives and the class of bounded derivatives by means of their associated sets. While he did not solve the problem he obtained a great deal of information about the delicate structure of derivatives. He points out that no such characterization is available for bounded derivatives and it was later observed (see [10, p. 100]) that the same is true for finite derivatives.

In a series of papers beginning with [55], Maximoff succeeded in characterizing various classes of functions related to derivatives via systems of perfect sets contained in the associated sets. This did not solve the characterization problem for derivatives; Agronsky [2] showed that this method was essentially equivalent to the associated set method and so must fail.

An entirely different approach was advanced by Neugebauer [58] for this problem. One observes (with Young of course) that a derivative is a function of the first Baire class that satisfies the Darboux property but that not all such functions are derivatives (cf. [WG 1910viii, p. 339]). What third condition, in the presence of these two conditions, describes derivatives? The analogous situation for integrals is well known: every integral is both continuous and has bounded variation, but this is not enough for a function f to be an integral. The third condition, provided by a theorem of Banach and Zarecki, is that f satisfy Luzin's condition N, i.e. that f maps null sets to null sets. Neugebauer finds such a condition which when added to Darboux, Baire 1 offers a characterization of derivatives ; unfortunately it involves additivity of interval functions and cannot be interpreted in terms of intrinsic properties of point functions in the spirit of the problem. The problem posed by Young thus remains open and, if anything, is even more interesting now.

§5. **Products of derivatives.** The article [W 1911vii], in addition to identifying the problem of the characterization of derivatives, identifies a class of related problems. The final sentence of the quote we have given points out that it is unknown when a product of two derivatives itself a derivative. The article then continues

> In view of the importance of the problem in the theory of the differentiation of infinite series and improper integrals—where the property of being a differential coefficient presents itself naturally as the necessary and sufficient condition which must hold in order that such a differentiation may, on certain assumptions, be permissible—as well as in other applications, it is hoped that the results here obtained will be regarded as of interest.

He summarizes his main results giving conditions when a product fg is a derivative and then asserts

> It will be seen that only the fringe of the main question of finding sufficient conditions that a function may be a differential coefficient has been touched. Partial, however, as the results obtained may from this point of view appear, they would seem from the nature of the reasoning employed to have a certain finality. Examples illustrating the limitations imposed by our theorems are desirable, but the work of constructing such examples is both difficult and laborious.

Young did not obtain any deep theorems in this area and it seems clear that he recognized the difficulties inherent in obtaining definitive results.

As an indication of some of the subtleties inherent in these problems, consider a product Fg' where F is differentiable and g' is a derivative. The regularity of F might cause one to expect that the product Fg' is a derivative. [W 1911vii, Theorem 3] shows that if F' and g' are summable, then this is the case. We now know that it is enough for *either* of these to be summable, but it is not enough to assume only that Fg' is summable. Even if not summable the product Fg' behaves very much like a derivative: there must be a dense open set on each component of which it is a derivative [3]. The following simple example shows that no more can be said. Let $F(x) = x^2 \sin x^{-3}$, $G(x) = x^2 \cos x^{-3}$, $F(0) = G(0) = 0$. One calculates readily that FG' and $F'G$ are both bounded, and so summable, and that $F(x)G'(x) - F'(x)G(x)$ is 3 if $x \neq 0$ and 0 if $x = 0$. If either FG' or $F'G$ were a derivative then so too would be the other since $(FG)' = FG' + F'G$. But the difference cannot be a derivative since it violates the Darboux property.

A related problem, that of determining necessary and sufficient conditions on a function F so that Fg' be a finite derivative whenever g' is, has been solved by Fleissner [28], based on work of Foran [29]. The analogous problems for certain subclasses of derivatives, for example the summable derivatives, have also been solved. See Mařík [50] and [51].

Recent research has focused on a number of questions related to the algebra generated by derivatives. The exposition [14] has an account; we limit ourselves here to a brief discussion that indicates further the delicacies faced in dealing with Young's problem.

Since products of derivatives need not be derivatives, what is the class of functions representable as a product of derivatives? This problem is as elusive as the problem of characterizing derivatives. For example a characteristic function of a closed set can be so written but a characteristic function of a (nontrivial) open set cannot [53]; these are special cases of deeper results that can be found in [54]. Any Baire 1 function that vanishes a.e. is such a product [13]. (This gives a quick proof [14] of the existence of differentiable, nowhere monotonic functions, a problem that has produced many incorrect examples starting with Volterra in 1887.) The most advanced theorem in this connection asserts that every Baire 1 function f can be written as $f = G' + H'K'$ (Preiss [64]); thus every Baire 1 function differs from a product of derivatives by a derivative.

Incidentally, representations of the form $f = G' + HK'$ where G, H and K are differentiable (note that this is stronger than the representation $f = G' + H'K'$ above) are possible for many classes of functions related to differentiation theory. For example approximate derivatives ([3]) and Peano derivatives ([26]) admit such a representation.

For additional recent results related to the representation of functions by combinations of derivatives see [54], [52]. For problems arising directly from Young's paper and an account of his contributions in that paper see the survey of Fleissner [27]; this also contains a correction of an oversight in [W 1911vii, Theorem 5].

§6. **Symmetry theorems.** In [W 1908i] is given the following theorem: if f is an arbitrary function of a real variable then for all values of x, excepting perhaps for a countable set,

$$\limsup_{h\to 0+} f(x+h) = \limsup_{h\to 0+} f(x-h) \quad \text{and} \quad \liminf_{h\to 0+} f(x+h) = \liminf_{h\to 0+} f(x-h).$$

Because the theorem was announced at the meeting of the British Association at Leicester in 1907 they used to refer to this as the "Leicester theorem." The next year at the Rome congress of 1908 this was improved to the statement that again for all values of x, excepting perhaps for a countable set all of the left and right limit numbers are identical. Stated in more modern language this theorem (naturally called the "Rome theorem") asserts that at all but countably many points the right and left cluster sets of an arbitrary function f are identical: *for an arbitrary real function* f

$$C^-(f,x) = C^+(f,x)$$

except at countably many points x. Here $C^+(f,x)$ denotes the set of all real or infinite numbers obtained as limits of some sequence $f(x_n)$ with $x_n \searrow x$, and $C^-(f,x)$ is the corresponding left hand version.

They went on to show in [W 1907ii] that the value $f(x)$ lies in the cluster sets, excepting again for countably many points, i.e. $f(x) \in C^+(f,x) = C^-(f,x)$. Similar themes, including analogous results for functions of several variables, reappear in the later papers [W 1910iii], [WG 1918i] and [W 1928iv] which is one of their last papers. (The exceptional set in the several variable case will no longer be countable.)

They had no small measure of (justified) pride in these theorems; for one thing they are genuinely interesting theorems that make an assertion true for a completely arbitrary function and they are perhaps the first such assertions. This also started a research program of searching for "asymmetry" results, i.e. results about the distinction between left and right as regards limits and derivatives. It is this line of research that culminated in the Denjoy–Young–Saks theorem and that has since generated dozens of related inquiries.

It is the foundation too for the large literature of cluster sets. About this theory Collingwood [20] remarks:

> The theory of the cluster sets of arbitrary real functions originated with W. H. Young. The story begins with his paper [W 1908i] in which he showed that the points of inequality of right and left upper and lower limits of a function of a single variable are enumerable. This was followed by a number of papers, some in collaboration with G. C. Young, of which the most important are [W 1910iii] in which he proved that for a function of a single variable the points of inequality of right and left cluster sets, although not under that or any other compendious name, are enumerable, with analogous theorems for several variables; and [W 1928iv] which completes and summarizes his theory. Young considered only real functions and was evidently unaware of Painlevé's definition of a cluster set *(domain d'indétermination)* which had been formulated in 1895 for complex functions. Perhaps for lack of a suitable terminology and notation to give point to the ideas Young's theorems

> attracted little notice and, so far as I can discover, have not hitherto been mentioned by writers on complex function theory. I am myself indebted to his daughter, Dr. R. C. H. Tanner, for calling my attention to them. The work of H. Blumberg (Fund. Math., 16 (1930) and 32 (1930)) who had independently discovered Young's theorem of 1908 on discontinuities (Blumberg, Bull. A. M. S. 24 (1918)) developed Young's theory of arbitrary real functions a good deal further and gave rise to theorems of Jarník (Fund. Math. 27 (1936)) and Bagemihl (P.N.A.S. 41 (1955)) whose well known ambiguous point theorem has important implications for complex function theory.

For more information on where the cluster set story has gone see the survey article of Belna [7] (who also quotes Collingwood) and the text [21].

Even just for real functions there remains considerable activity in the study of cluster sets. The first natural way to generalize these concerns is to replace ordinary limits by approximate limits, that is to say to use the density topology rather than the usual topology of the reals. The first study appears to have been that of Kempisty [44] who proves a very weak approximate version of the Young's early results: *for an arbitrary real function f*

$$\sup C_e^-(f,x) \geq \inf C_e^+(f,x) \quad \text{and} \quad \sup C_e^+(f,x) \geq \inf C_e^-(f,x)$$

except at countably many points x. Here $C_e^+(f,x)$ generalizes the ordinary right hand cluster set $C^+(f,x)$; it denotes the set of all real or infinite numbers obtained as limits of $f(x+h)$ with $h \searrow 0$, and h in some set H having positive upper right exterior density at 0. $C_e^-(f,x)$ is the corresponding left hand version.

Except for this early result of Kempisty the essential cluster sets were not much studied. It was not until Zahorski asked whether the approximate analogues of the Young's cluster set theorems should be true that a series of studies began to appear. Answers to this question can be found in the articles of Belowska [8], Bruckner and Goffman [12], Goffman [36], Goffman and Sledd [37], Kulbacka [45], O'Malley [59] and others.

The activity in this area continues even today. See [72, Chapter 2] for a general treatment of real cluster sets in an abstract setting following ideas developed by the school at Łódź (Świątkowski, Jedrzejewski and Wilczyński).

§7. **Denjoy–Young–Saks Theorem.** The origins of the Denjoy–Young–Saks Theorem can be found in early work of the Youngs, particularly in the cluster set contributions discussed in the previous section. In a footnote to her paper [G 1916v] she reports:

> I would recall the theorem communicated by my husband to the British Association at Leicester in 1907, and in an extended form, to the Mathematical Congress at Rome in 1908, that, except at a countable set of points, the limits of $f(x)$ on the right and on the left are the same and $f(x)$ lies between the greatest and the least of these limits (inclusive) on each side. ... These surprising theorems formed the starting point of our investigations, and suggested

that similar general results could be obtained for derivates: till now, however, we had not been able to justify such a supposition.

The theorem that she cites here is what they called the "Rome theorem" and which we have already discussed. The program was rather evident: having obtained such remarkable results for completely arbitrary functions as regards one-sided limits one fully expects there to be equally interesting results for one-sided derivates.

The first result in this program as applied to derivatives is [W 1908xi]: *if f is continuous then everywhere except at a set of points of the first category*

$$f^+(x) = f^-(x) \quad and \quad f_-(x) = f_+(x).$$

In fact this theorem was available as a corollary to the study of limits of semicontinuous functions but he gives a direct proof too ([W 1908xi, p. 306]). As he remarks this should be compared with the Lebesgue differentiation theorem for functions of bounded variation whose derivates must agree outside a set of measure zero. They evidently expected (cf. [G 1916v, p. 36]) that this set too should have measure zero. Of course this is false (a counterexample appears in Denjoy [23, §59, Ex. (III)]) but this expectation kept them looking in the right direction.

The program was continued in [G 1914ii] where it is proved that *for an arbitrary function f everywhere except at a countable set of points*

$$f^+(x) \geq f_-(x) \quad and \quad f^-(x) \geq f_+(x).$$

At the time that this appeared there had already been a number of closely related theorems (due to Hilbert, Levi, Rosenthal and Sierpiński) but stated and proved under much narrower assumptions. (This theorem is proved in [68, p. 262] along with some other related results.)

She soon turned to the much deeper and more important problem of determining what relations should hold with the exception of a set of measure zero. In [G 1916iii], [G 1916v] and [G 1916vi] she presents her findings. At the same time on the continent Denjoy [23] had attacked the identical problem. She and he independently discovered the relations that must hold for continuous functions, and she proved as well that they hold for an arbitrary measurable function. The arguments needed to remove the hypothesis of measurability were supplied later by Saks [67]. In this way we come to the statement that is known as the Denjoy–Young–Saks theorem[5]: *for an arbitrary real function f every point x with the exception only of a set of measure zero falls into one of the following four sets*

1. *A_1 on which f has a finite derivative,*

2. *A_2 on which $f^+(x) = f_-(x)$ (finite) $f^-(x) = +\infty$, $f_+(x) = -\infty$,*

[5]Perhaps [39] is the first publication to so label the theorem.

3. A_3 on which $f^-(x) = f_+(x)$ (finite) $f^+(x) = +\infty$, $f_-(x) = -\infty$, and

4. A_4 on which $f^-(x) = f^+(x) = +\infty$ and $f_-(x) = f_+(x) = -\infty$.

(Each set A_1, A_2, A_3 and A_4 can have positive measure.)

An elementary proof may be found in [66, pp. 18–19]. An account of a number of contemporary related papers appears in [68, p. 271]. In [G 1922ii] she returns to the theorem in the setting of functions of two variables and gives another proof.

Even just for the ordinary Dini derivatives more can be said. In a series of papers Garg [31], [32], [33], [34] and [35] extends the Denjoy–Young–Saks theorem and gives a variety of applications. Another interesting variant was discovered by Evans and Humke [24]. If f is monotonic then rather sharper statements on the exceptional sets can be made. Of course f is almost everywhere differentiable but the relation between the Dini derivatives is even tighter: *if f is monotonic then the set of points x where $f^-(x) \neq f^+(x)$ or $f_-(x) \neq f_+(x)$ is σ–porous.* The class of σ–porous sets is smaller than the class of measure–zero, first category sets. See [72, p. 158, pp. 160–161]) for many further extensions.

The success of the Denjoy-Young-Saks theorem in classifying the relations that hold among the four Dini derivatives has prompted a similar study in almost every other setting where the ideas make sense. The most important and useful generalization is in the setting of the approximate derivative but there are numerous other contexts in which the ideas can be usefully applied. For the approximate Dini derivatives a completely analogous situation holds as for the original Denjoy–Young–Saks theorem. The original work is due to Burkill and Haslam-Jones [16] and to Ward [74]; later clarifications and improvements can be found in Zajíček [79]. For symmetric derivates see [30]; for congruent derivates [70]; for path derivates [15]; for negligible derivatives [22]; and for qualitative derivates [25].

A further account of this material and a general perspective on it may be found in [72, Ch. 6, 7].

§8. **Differentials** In Hardy's account [40, p. 232] of Young's contributions he makes a particular mention of the work on differentials. Certainly Young's best work is in Fourier series, integration theory, cluster sets and differentiation but, as Hardy puts it, his most *important* work is perhaps in this elementary treatment of differentials and implicit functions.

The articles [W 1909vii], [W 1909ix], [W 1909xi] and the Cambridge Tract of 1910 set out this theory in a clear and lucid way. Hobson's treatment of this subject in [43] is almost entirely due to Young as Hobson acknowledges fully. By now these ideas have worked their way into countless university calculus courses and the sources have disappeared entirely.

In the nineteenth century presentations the differential of a function $f(x, y)$ was defined as $f_x dx + f_y dy$ without any further requirements. At the end of the century Stolz had presented a definition of differentiability that we would now recognize as correct but he had not seen how this should develop. Young's presentations develop on this definition, treat the equality of the mixed partials, introduce higher order differentials and handle implicit functions in an elegant and simple manner.

It is interesting to note that the final paper of their career [G 1930i] again exploits the development of differentials. The intention was to produce a real variable proof of the Cauchy-Goursat theorem. As Cartwright [17] points out in her obituary notice on Grace, there is an error in the paper (Theorem 5, p. 82) which invalidates the results. The error is easy to spot and in earlier years she would most certainly have not made this mistake, but by this time both of the Youngs had essentially retired from mathematics and they were not in good health.

§9. **Mean value theorems.** The Youngs [W 1909i] [WG 1909iii] discovered an elementary but interesting extension of the mean-value theorem of the calculus. Their theorem asserts that *if f is a continuous function on an interval $[a, b]$ such that there is no distinction of right or left with regard to the derivates of f then there is a point $\xi \in (a, b)$ with $f'(\xi) = (f(b) - f(a))/(b - a)$.* By applying this to subintervals they observe that in fact the derivative $f'(x)$ exists on a c–dense subset of $[a, b]$ and f' has the Darboux property on that set. Note that an easy monotonicity theorem follows too: if f has these properties and its derivative is positive at every point where it exists then f is increasing.

The "no distinction" phrase in the statement of the theorem merely means that the right and left pairs of Dini derivatives agree at each point. This class of functions includes the class of smooth functions studied by Zygmund [80]. A continuous function f is *smooth* if

$$\lim_{h \to 0+} h^{-1}(f(x + h) + f(x - h) - 2f(x)) = 0$$

at each point. It is easy to see that such a function possesses the property of the Youngs' theorem. Indeed for these functions this same theorem was rediscovered by Zygmund in 1945 (ascribing parts of it to Rajchman and Zalcwasser). In this instance, as in so many others, the Youngs fail to acquire any credit even though the theorem should have been widely known (for example it is cited in Hobson [43, Vol. I, p. 384]).

Though elementary and not difficult to prove the result has some considerable interest. For example with it and classical material on trigonometric series one can show immediately that the sum of the series

$$s(x) = \tfrac{1}{2}a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

with a_k, $b_k = o(1/k)$ has the Darboux property on the set where the series converges. In particular if this is an everywhere convergent Fourier series of a function f then f can have no discontinuities of the first kind. These considerations can also be used to show that in the original Youngs' theorem the set of points where the derivative exists, while it is c–dense in every interval, may have measure zero. These observations are from [80].

Apparently the Youngs did not apply these ideas to Fourier series. They do apply them in further papers ([W 1909v], [W 1909viii], [W 1909i]) to Taylor's theorem, L'Hôpital's rule and monotonicity theorems.

Generalized versions of the mean-value theorem continue to play a role. This theme can be continued in various ways. What conditions on a function ensure that it must have a derivative on some substantial set and possess the Darboux property there? What generalized derivatives possess forms of a mean-value theorem?

While the literature pursuing such questions is rather large we can give some flavour of recent results by reporting on one direction. Let f be approximately differentiable. It was long known [73] that f' must exist on an open dense set. A series of papers [60], [75], [61] showed that f' has the Darboux property; in fact if f'_{ap} takes the values a and b ($a < b$) on an interval then on some subinterval f' exists and assumes all values in $[a, b]$. Generalized versions of this property can be found in [15] and [71].

As would be expected such theorems have applications to monotonicity theorems of various sorts. For example any monotonicity theorem valid for differentiable functions must have counterparts for functions differentiable in more general senses (meeting a variety of hypotheses). As a particular instance (again in the setting of the approximate derivative) one can conclude that if f is approximately differentiable and monotonic on any interval on which it is differentiable then f is monotonic. (Note that the negative of the Cantor function provides counterexamples for many similar, but misguided, conjectures.)

§10. **Further commentary.** Our account is far from a complete representation of the range of topics in real variable theory that the Youngs addressed. Let us mention briefly some further types.

A topic that was reasonably in vogue at the time was the study of the continuous nowhere differentiable functions that had been introduced by Bolzano, Cellérier and Weierstrass. While some authors scorned their study (the preface to [68] contains some famous quotations in this regard) the Youngs evidently considered them a legitimate object of investigation. [W 1908xv], [WG 1911v] address this and [G 1916v, pp.376–377] gives a detailed report. (The account in Hobson [43, Vol II, pp. 402–412] is entirely theirs.) In particular they show that the Weierstrass function does have one-sided vertical tangents at many points. It is not until Besicovitch [9] that an example of a continuous function without a tangent in any sense is given.

Curiously these "irregular" curves are today much more in fashion than ever because of the interest of the fractalists. Indeed the flamboyant language of Grace Young in her Gamble prize essay [G 1916iii],

> Away with your ordinary curves, the wild atom will have none of them

expresses quite a modern sentiment. Less poetically, but with remarkable insight, she suggests

> I cannot but think that these curves will serve as the basis of the geometrical theory of molecular phenomena, in the same sense as the conic sections have served as a first approximation to the movements of the planets.

The essay is beautifully and carefully written and is a good source of historical information on this topic. It reveals too the contemporary attitude of their colleagues as regards such subjects; one anonymous, but presumably eminent, Cambridge mathematician is quoted as saying "No one can take any real interest in derivates."

In their study of derivates the Youngs soon realized that some kind of generalized derivative was needed because in many investigations one does not have ordinary derivatives available and the ordinary derived numbers are of no particular use. They were aware of the Riemann's second symmetric derivative which plays a role in the theory of trigonometric series but apparently they had no other ideas. The only positive suggestion that they made in this regard is the "mean symmetric derivative" that appears in [G 1914ii] but this has not survived as a useful tool (as is acknowledged in [W 1926iii]). Since then numerous generalized derivatives have been proposed and studied. The most important of these is the approximate derivative which shares most of the properties of the ordinary derivative. We have mentioned (see §3, §5, §6, §7, §9) that many of the Youngs' results extend to approximate derivatives and other generalized derivatives. Theorems such as the one we mentioned at the end of §5 offer some insight here: since the representation $F'_{ap} = G' + HK'$ is valid for each approximate derivative we see that F'_{ap} is "almost" a sum of two derivatives and hence almost itself a derivative. In fact ([14]) if H' is locally summable then F'_{ap} *is* a derivative and, in any case, F is differentiable on a dense open set.

A number of articles ([W 1907ii, [W 1909ii], [WG 1910viii], [W 1916iii]) are devoted to a deep study of the properties of semicontinuous functions. Of course now semicontinuity is regarded as equally important as the notion of continuity; at the time of their writing there had been scarcely any attention paid. In fact, as Young points out in [W 1909ii],

> The semicontinuous functions introduced by Baire [in 1899] have not perhaps received the amount of attention which their importance and usefulness might appear to have justified. With the exception of their originator, I cannot call to mind any one beside myself who has utilised them to any great extent.

It is hard to imagine much of modern analysis without the use of such functions and, to some degree, Young should be given credit for promoting and exploiting these ideas. [W 1910ii, p. 106], [W 1913ix, p. 213], [G 1914ii, p. 151] and [W 1916i, pp. 49–51] give monotonicity theorems employing semicontinuity conditions that extend the classical theorem of Scheefer.

In [W 1907ii] Young provided a useful condition for determining whether or not a Baire 1 function f has the Darboux property. One must check only whether, for all x, the value $f(x)$ belongs to both the right and left cluster set at x. This provides the first of many characterizations of the Darboux property within the class of Baire 1 functions (see [10, Theorem 1.1]). Some of these follow readily from Young's. Conditions such as these are important because of the relevance of the class of Darboux, Baire 1 functions to differentiation theory. This class contains most generalized derivatives as well as their primitives and every such function is topologically equivalent to a derivative. Extensions to functions of several variables and to mappings of various spaces are numerous, the form of the extension depending on the spaces under consideration and the type of "Darboux property" under consideration. (The article [W 1909xvi] discusses the two variable case, but it is dominated by a lengthy footnote devoted to continuing an attack on Schoenflies. For a discussion of this bitter and long lasting dispute between Young and Schoenflies see [19].)

All references to the works of G. C. Young and W. H. Young in the essay are given in a format allowing use of the bibliography in the Selected Papers, *Presses Polytechniques et Universitaires Romandesin (2000), edited by S. D. Chatterji and H. Wefelscheid [1]. That bibliography in turn was based on a definitive one given by the historian I. Grattan-Guiness in* Historia Mathematica *2, (1975), 43–58. Thus for example [G 1914ii] indicates a paper by Grace that would have been published in 1914 and represents the second paper that the pair published, separately or together, in that year. Similarly [W 1903ii] indicates a paper by her husband, and [WG 1909iii] a joint publication. The Grattan-Guiness scheme was similar but the chronology is occasionally different as is explained by the editors.*

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