which establishes (3). Equality holds if $A_i = \frac{1}{n}$ since $\phi_2(x)$ is strictly log-convex.

Now let $\phi_3(x) = (1-x)/x$; then $\phi_3(x)$ is log-convex for $0 < x \leq \frac{1}{2}$ and log-concave for $\frac{1}{2} \leq x < 1$. Also let $\mathbf{A} = (A_1, \dots, A_n)$, where $0 \leq A_i \leq \frac{1}{2}$. Clearly

$$\mathbf{A} \succ \left(\frac{\Sigma A_i}{n}, \dots, \frac{\Sigma A_i}{n} \right),$$

so that

$$\prod_{i=1}^{n} \left(\frac{1-A_i}{A_i} \right) \geq \left[\frac{1-\Sigma A_i/n}{\Sigma A_i/n} \right]^n = \left[\frac{n-\Sigma A_i}{\Sigma A_i} \right]^n,$$

establishing (4).

Finally, let $\mathbf{A}^1 = ((n-1)A_1, \dots, (n-1)A_n)$ and let $A^2 = (1-A_1, \dots, 1-A_n)$ where $A_i \geq 0, \Sigma A_i = 1$. It can be easily verified that $\mathbf{A}^1 \succ \mathbf{A}^2$. By Theorem 1, $g(\mathbf{x}) = \Pi_{i=1}^{n} x_i$ is a Schur-concave function. It then follows that

$$(n-1)^n \prod_{i=1}^{n} A_i = \prod_{i=1}^{n} (n-1)A_i \leq \prod_{i=1}^{n} (1-A_i),$$

proving inequality (2).

It is apparent that many additional inequalities of the Weierstrass product type can be formulated and proved by choosing the appropriate log-concave function, forming products to obtain a Schur-convex function, and then using Definition 2 above.

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CLASSROOM NOTES

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REGULATED FUNCTIONS: BOURBAKI'S ALTERNATIVE TO THE RIEMANN INTEGRAL

S. K. BERBERIAN

1. Introduction. At the outset, I hasten to say that I remain a "Riemann loyalist": pound for pound, the Riemannian circle of ideas can't be beat for its instructional value to the beginning student of analysis. Consequently, I wouldn't go so far as to suggest that the theory of regulated functions replace the Riemann integral in the beginning undergraduate analysis course; however, in a graduate course in real variables, the theory of regulated functions can be an entertaining alternative to a routine review of the Riemann integral; and it is in some ways a more instructive prelude to the Lebesgue theory, as I hope to persuade the reader in this brief "comparative anatomy" of integration theories.

2. Regulated functions. In the following, $[a,b]$ denotes a fixed, nondegenerate closed interval of the real line $\mathbf{R}$, and f, g, $F,\ldots$ are real-valued functions on $[a,b]$, assumed to be bounded when they need to be bounded.

In the class of Riemann-integrable functions, Exhibit A is the class of continuous functions; Exhibit B is the class of monotone functions (and its linear span, the functions of bounded variation). There is a property shared by these two special classes of functions: at every point of the interval, the function possesses finite one-sided limits. Such a function is said to be *regulated.* (A rationale for the terminology: such a function is "limited on the left" and "limited on the right." The classical term: function with only discontinuities of the "first kind.") The regulated functions form an algebra of functions for the pointwise operations, by the algebraic properties of limitibility (in other words, by the continuity of the algebraic operations on real numbers). Every regulated function is bounded (by an easy contradiction argument based on the Weierstrass–Bolzano theorem). The uniform limit of regulated functions is regulated, by the "iterated limits theorem" [**5**, p. 149, Th. 7.11]. From the viewpoint of integration theory, the most transparent example of a regulated function is a *step function*, that is, a function with finitely many values, each assumed on an interval, possibly degenerate (in other words, a linear combination of characteristic functions of intervals, possibly degenerate). Every uniform limit of step functions is regulated. In fact, there are no other regulated functions: every regulated function is the uniform limit of step functions (by essentially the same argument, based on the Heine–Borel theorem, used to show that a continuous function is uniformly approximable by step functions) [**1**, Ch. 2, §1, no. 3, Th. 3]; [**2**, p. 139, Th. 7.6.1]. A step function f comes very close to having a continuous anti-derivative: there exists a continuous function F (necessarily piecewise linear) whose derivative exists and is equal to $f(x)$ except at the finitely many points of discontinuity of f; suggestively,

$$\exists F'(x)=f(x) \text{ f.e.},$$

where "f.e." means "with finitely many exceptions." This implies, by a standard theorem on term-by-term differentiation [**5**, p. 152, Th. 7.17], that for every regulated function f, there is a continuous function F whose derivative exists and is equal to $f(x)$ for all but countably many values of x; suggestively,

$$\exists F'(x)=f(x) \text{ c.e.},$$

where "c.e." means "with countably many exceptions."

3. An elementary integration theory. The foregoing discussion suggests this definition: call F a *primitive* of f if (1) F is continuous, and (2) $\exists F'(x)=f(x)$ c.e. (More precisely, one could say that F is a "c.e.-primitive" of f. If $F'(x) = f(x)$ for all x, one calls F a *strict* primitive of f; we remark that the range of f must then be an interval [**5**, p. 108, Th. 5.12].) As noted above, every regulated function has a primitive. (The converse is false; see Section 6 below.)

If f has a primitive F, one is tempted to define

$$\int_a^b f = F(b)-F(a).$$

This is a legitimate definition, since any two primitives of f must differ by a constant, by virtue of the following result [**1**, Ch. 1, §2, no. 3, Cor. of Th. 2]: If H is continuous and if $\exists H'(x)=0$ c.e., then H is constant. (Indeed, it suffices to suppose that H is continuous and that $\exists H_r'(x)=0$

c.e., where H_r' denotes right derivative.) One has here the makings of an elementary theory of integration: call a function $f:[a,b]\to\mathbf{R}$ *integrable* if it has a primitive F in the above sense, and then define $\int_a^b f = F(b)-F(a)$. The integrable functions form a linear space (but not a linear algebra, as noted in Section 6); the uniform limit of integrable functions is integrable, by the theorem on term-by-term differentiation mentioned earlier. In this integration theory, the class of "primitives" is the class of functions $F:[a,b]\to\mathbf{R}$ such that (1) F is continuous, and (2) $\exists F'(x)$ c.e. Examples: F continuous and monotone; more generally, (i) F continuous and of bounded variation [**6**, p. 107], or (ii) F continuous and *convex* [**1**, Ch. 1, §4, no. 4, Prop. 8]. (Incidentally, if f is monotone (hence f has a primitive), then every primitive of f is convex [**1**, Ch. 2, §1, no. 3, Prop. 4].)

REMARK. If f is integrable and $g=f$ c.e., then g is integrable (with the same primitives as f); this follows trivially from the definitions.

4. The Lebesgue integral. Lebesgue's "fundamental theorem of calculus" gives a succinct characterization of Lebesgue-integrability [**6**, pp. 198, 201]: A function f is Lebesgue-integrable if and only if there exists an *absolutely continuous* function F such that $F'(x)=f(x)$ a.e.; one then has $\int_a^b f(x)\,dx = F(b)-F(a)$. Relevant here is the following fact: If H is absolutely continuous and $H'(x)=0$ a.e., then H is constant [**6**, p. 205]. (Incidentally, one cannot weaken "absolutely continuous" to "continuous of bounded variation," as is shown by Lebesgue's famous example of a nonconstant continuous increasing function whose derivative vanishes almost everywhere [**3**, p. 96].) In Lebesgue's theory, the class of "primitives" is the class of absolutely continuous functions (such a function always possesses a derivative a.e.).

REMARK. If f is Lebesgue-integrable and $g=f$ a.e., then g is Lebesgue-integrable (with same "absolutely continuous, a.e.-primitives" as f).

5. The Riemann integral. Here f denotes a bounded function, D_f its set of points of discontinuity. The succinct criterion for Riemann-integrability is that of Lebesgue: f is Riemann-integrable if and only if D_f is Lebesgue-negligible [**6**, p. 142]. Then the formula $F(x)=\int_a^x f(t)\,dt$ defines an absolutely continuous (indeed, Lipschitz) function, with $F'(x)=f(x)$ a.e. and in particular for $x\notin D_f$.

The class of "primitives" F for this theory is somewhat clumsy to describe: (1) F is Lipschitz (hence absolutely continuous, with bounded derivative), and (2) there exists a bounded function $f:[a,b]\to\mathbf{R}$, continuous a.e., such that $F'(x)=f(x)$ a.e.

The Riemann-integrable functions form an algebra of functions, closed under uniform limits; this is easy to see, for instance, from Lebesgue's criterion.

REMARK. If f is Riemann-integrable and $g=f$ c.e., it does not follow that g is Riemann-integrable. (For example, let f be the function identically zero, g the characteristic function of the set of rational numbers in $[a,b]$.) Of course all is well if "c.e." is replaced by "f.e." Thus, in a sense, in the Riemann theory the natural "negligible" sets are the finite sets; and, in this sense, the "c.e." theory described in Section 3 above is more flexible.

If f is continuous c.e. and is bounded, then f is Riemann-integrable (by Lebesgue's criterion) and the function $F(x)=\int_a^x f(t)\,dt$ is a c.e.-primitive of f (and is absolutely continuous—even Lipschitz). Example: f any regulated function [**1**, Ch. 2, §1, no. 3, Th. 3].

6. Miscellaneous examples. Let us write C for the class of continuous real functions on $[a,b]$, BV for the functions of bounded variation, L for the regulated functions, R for the Riemann-integrable functions, L^1 for the Lebesgue-integrable functions, and I for the class of functions "integrable" in the sense of Section 3. One has the diagram, shown in Figure 1, where the lines represent inclusion relations (all of them proper, as we shall see).

$L\subset R$ properly: Let (x_n) be a sequence in $[a,b]$, $a<x_n\leqslant b$, such that $x_n\to a$, and let f be the

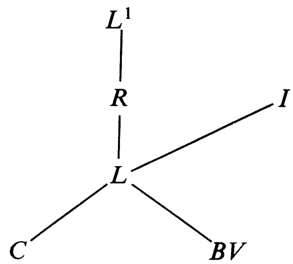

FIG. 1

characteristic function of the set $S=\{x_1,x_2,x_3,\dots\}$. The set of discontinuities of f is S, thus $f\in R$; however, $f(a+)$ does not exist, thus $f\notin L$.

$L\subset I$ properly: If one had $L=I$, then I would be an algebra of functions; thus, it will suffice to exhibit functions f, g in I such that $fg\notin I$. Let [**1**, Ch. 2, §2, Exer. 4] f and g be functions on $[a,b]$ that possess strict primitives, and for which there exists a continuous function $H:[a,b]\to\mathbf{R}$ such that (i) $\exists H_r'(x)=f(x)g(x)$ on $[a,b)$, and (ii) the set $\{x: H$ is not differentiable at $x\}$ is uncountable. It follows that fg has no primitive; for, if K were a primitive of fg, one would have $(H-K)_r'(x)=0$ c.e.; then H would differ from K only by a constant, which contradicts property (ii). (Incidentally, consideration of the identity $fg=\frac{1}{2}[(f+g)^2-f^2-g^2]$ shows that there exists a function h such that h admits a strict primitive but h^2 admits no primitive.)

It is straightforward to see that the remaining inclusions in the diagram are proper. (Here is a devious way of seeing that $R\subset L^1$ properly: If one had $R=L^1$, then L^1 would be an algebra of functions; then $f\in L^1$ would imply $f^2\in L^1$; that is, $f\in L^2$; but $L^1\not\subset L^2$ (look at the function $x^{-1/2}$ in the unit interval).)

No inclusion relation exists between R and I, as the following remarks show.

$I\not\subset R$: Indeed, there exists a function that possesses a strict primitive but is not Riemann-integrable [**3**, p. 43].

$R\not\subset I$: Let f be the characteristic function of the ternary Cantor set in $[a,b]=[0,1]$. It is easy to see that the set of discontinuities of f is the Cantor set, consequently $f\in R$. However, $f\notin I$. For, if f had a primitive F in the sense of Section 3, one would have $F(x)=\int_0^x f(t)\,dt+F(0)$ for all x [**4**, p. 299, Exer. 18.41(d)]; since $f(t)=0$ a.e., this means that F is constant. Then $F'(x)=0$ for all x; but, by hypothesis, $F'(x)=f(x)$ for all but countably many values of x, consequently $F'(x)=1$ for uncountably many x, a contradiction.

Finally, there is no inclusion relation between I and L^1. On the one hand, $L^1\not\subset I$ (better yet, $R\not\subset I$). On the other hand, let F be a continuous function on $[a,b]$ such that $F'(x)$ exists on (a,b) but is not Lebesgue-integrable [**4**, p. 299, Exer. 18.42]; if f is the function such that $f(x)=F'(x)$ on (a,b) and (say) $f(a)=f(b)=0$, then $f\in I$ but $f\notin L^1$.

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