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THE EQUATION $u_x u_y=0$ FACTORS

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In a recent correspondence with one of the authors, Lee Rubel asked whether every solution (on the plane $\mathbf{R}^2$) of the partial differential equation

$$\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}=0$$

must be a function of one variable. For solutions in $\mathscr{C}^2$, the question is easily answered: differentiate $u_x u_y=0$ with respect x and y, then multiply these equation by u_x and u_y respectively, we obtain after addition $(u_x^2+u_y^2)u_{xy}=0$. If in a point $p\in\mathbf{R}^2$ we had $u_{xy}\neq 0$ then $u_x^2+u_y^2=0$ would imply $u=\text{const}$, thus one finds that a solution must satisfy $u_{xy}=0$ on $\mathbf{R}^2$, whence u is of the form $u(x,y)=f(x)+g(y)$ and $\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}=f'(x)g'(y)$. If $g'(y_0)\neq 0$, then $f'\equiv 0$, so $u(x,y)=g(y)+\text{constant}$. In a later correspondence Rubel mentioned that W. Jockusch had obtained an affirmative answer under the assumption that $u\in\mathscr{C}^1(\mathbf{R}^2)$.

The purpose of the present note is to show that Rubel's question has an affirmative answer whenever the equation makes sense; that is, whenever both partials of u exist on all of $\mathbf{R}^2$. In fact, our theorem shows a bit more. If u is continuous in each variable separately and at each point in $\mathbf{R}^2$ one of the partials vanishes, then u is a function of one variable. We do not assume that both partials exist at every point.

Our method is to first establish the result under the assumption that u is continuous and then to show that the hypotheses of our theorem actually imply continuity.

LEMMA 1. *Let u be continuous on a neighbourhood of a closed rectangle $R\subset\mathbf{R}^2$, let p be the lower left vertex of R, and let C be the component of the set $\{q\in R;\ u(q)=u(p)\}$ containing p. Suppose that at each point of R at least one of the partial derivatives exists and vanishes. Then C intersects at least one of the two edges of R not containing p.*

PROOF. If C does not intersect either of the two edges of R not containing p, we use the compactness of $\{q\in R;\ u(q)=u(p)\}$ to find disjoint relatively open subsets U and V of R such that $p\in U$, the union of the two edges of R not containing p is a subset of V, and $\{q\in R;\ u(q)=u(p)\}\subset U\cup V$. (This follows, for example, from

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the equality of components and quasi-components in compact spaces. See [1, § 47, II, Theorem 2].) Since u is continuous and R is compact, we may find $\varepsilon>0$ such that $|u(q)-u(p)|\geqq 2\varepsilon(|q_1-p_1|+|q_2-p_2|)$ for every $q\in R\setminus(U\cup V)$. Let r be the largest point in the lexicographic order of the set $\{q\in\overline{U};\ |u(q)-u(p)|\leqq\varepsilon(|q_1-p_1|+|q_2-p_2|)\}$. Since $|u(s)-u(p)|\geqq 2\varepsilon(|s_1-p_1|+|s_2-p_2|)$ for every $s\in\overline{U}\setminus U$, r belongs to U. Consequently, $|u(q)-u(r)|\geqq\varepsilon(|q_1-r_1|+|q_2-r_2|)$ whenever q is sufficiently close to r, $q_1\geqq r_1$, and $q_2\geqq r_2$. But this contradicts the assumption that $u_x=0$ or $u_y=0$.

LEMMA 2. *Suppose u, R, and p satisfy the conditions of Lemma* 1. *Then the value of u at one of the corners of R adjacent to p is equal to $u(p)$.*

PROOF. Let r be a corner of R adjacent to p such that the component C_0 of the set $\{q\in R;\ u(q)=u(p)\}$ containing p meets the edge not containing p and having r as one of the end points. Using Lemma 1 in a symmetric situation, we see that the component C_1 of the set $\{q\in R;\ u(q)=u(r)\}$ containing r meets at least one of the two edges of R not containing r. But then $C_0\cap C_1\neq\emptyset$, which immediately shows that $u(r)=u(p)$.

LEMMA 3. *Let u be a continuous function defined in an open rectangle. Suppose for each point of this rectangle at least one of the partials of u exists and vanishes. Then u is a function of one variable.*

PROOF. If u is constant on all vertical lines, the statement holds true. Thus suppose that there are two points p and q with the same abscissa and with different values of u. Then for every point r with the same abscissa either $u(r)\neq u(p)$ and we apply Lemma 2 to rectangles with two corners at r and p to deduce that u is constant on the horizontal line passing through r, or $u(r)\neq u(q)$ and we apply the same Lemma to rectangles with two corners at r and q.

LEMMA 4. *Let u be defined on the plane $\mathbf{R}^2$ and continuous in each variable separately. Suppose that at each point of the plane at least one of the partial derivatives exists. Then every nonempty closed set $P\subset\mathbf{R}^2$ contains a portion on which u is continuous.*

PROOF. Let $p\in P$. Since at least one of the partial derivatives of u at p exists, there is $n=1, 2, \ldots$ such that for every q in $\mathbf{R}^2$ with $|p-q|<1/n$ and with the same abscissa (or perhaps ordinate) the inequality $|u(p)-u(q)|<n|p-q|$ is satisfied. For each $n=1, 2, \ldots$ let A_n denote those points of P for which the above inequality holds with respect to the abscissas and B_n the corresponding set with respect to the ordinates. The Baire Category Theorem implies that for some n one of the sets A_n or B_n is dense in a portion Q of P. Suppose that it is A_n. To show that u is continuous on Q it suffices to prove that for each point $p\in Q$,

$$u(p)=\lim_{q\to p;\,q\in A_n} u(q).$$

Let $p\in Q$ and $\varepsilon>0$. Because u is separately continuous, there is $0<\delta<\varepsilon/(n+1)$ such that if r has the same abscissa as p and $|r-p|<\delta$ then $|u(r)-u(p)|<\varepsilon/(n+1)$. Let $q\in A_n$ and satisfy $|q-p|<\delta$. Let r be the point with the same abscissa as p and the same ordinate as q. Then $|u(q)-u(p)|\leqq|u(q)-u(r)|+|u(r)-u(p)|\leqq n|q-r|+\varepsilon/(n+1)\leqq n\varepsilon/(n+1)+\varepsilon/(n+1)=\varepsilon$.

THEOREM. *Let u be a function defined in* $\mathbf{R}^2$ *and continuous in each variable separately. Suppose for each point of* $\mathbf{R}^2$ *at least one of the partials of u exists and vanishes. Then u is a function of one variable, i.e.* $u_x=0$ *or* $u_y=0$ *identically.*

PROOF. By Lemma 3, it suffices to prove u is continuous. Let E be the interior of the set of continuity points of u. By Lemma 4 we know that E is dense in $\mathbf{R}^2$. We show $\mathbf{R}^2\setminus E$ is empty. If this were not so, there would be, by Lemma 4 an open square S such that $P=S\setminus E$ is nonempty and the restriction of u to P is continuous. We show in fact that u, as a function on $\mathbf{R}^2$ is continuous at each point of P, and this implies a contradiction immediately. Let $p\in P$ and $\varepsilon>0$. Let H and V be the horizontal and vertical lines through p, respectively. Then there is $\delta>0$ such that if $s\in P\cup H\cup V$ and $|s-p|<\delta$ then $|u(s)-u(p)|<\varepsilon$. Now let $q\in E$ satisfy the inequality $|q-p|<\delta$.

If u is not constant in any neighbourhood of q, by Lemma 3, u is a function of one variable, say the first, on every rectangle T satisfying $q\in T\subset E$. Let W be the vertical line through q. It follows from the assumption of separate continuity that there exists a segment $J\subset W$ containing q and a point $s\in S\cap W\cap(P\cup H)$ such that $|s-p|<\delta$ and u is constant on J. The inequalities

$$|u(q)-u(p)| = |u(s)-u(p)| < \varepsilon$$

establish the continuity of u at p.

In case u is constant in some neighbourhood of q there are two cases. Either there is $r\in S\cap(P\cup H\cup V)$, $|r-p|<\delta$ with $u(r)=u(q)$; in that case $|u(q)-u(p)|<\varepsilon$ or, there is $r\in S\cap E$, $|r-p|<\delta$ such that u is not constant in any neighbourhood of r and $u(r)=u(q)$. In that case we apply the previous argument to r and once again obtain $|u(q)-u(p)|<\varepsilon$.

Thus u is continuous on all of S and P is empty, a contradiction.

REMARKS. (i) It is easy to construct examples to show that the assumption of separate continuity cannot be dropped in the statement of the Theorem.

(ii) One can replace $\mathbf{R}^2$ by any rectangular region in the statement of the Theorem. The theorem fails, however, for any region that is not rectangular, even for $\mathscr{C}^\infty$ functions. On the other hand, any counterexample on a nonrectangular region must be constant on some set with nonempty interior.

(iii) Finally let us point out that there is no analogous result in higher dimensions. For example the function

$$f(x,y,z)=\begin{cases} x\exp(-z^{-2}), & \text{if } z>0\\ 0, & \text{if } z=0\\ y\exp(-z^{-2}), & \text{if } z<0\end{cases}$$

is in $\mathscr{C}(\mathbf{R}^3)$ and satisfies

$$\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\frac{\partial f}{\partial z}=0$$

showing that $u_x u_y u_z=0$ does not factor. Note that this example shows even that in dimensions higher than two the equation $u_x u_y=0$ does not factor.

Reference

[1] K. Kuratowski, *Topology II*. Academic Press, 1968.

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