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Real variable contributions of G. C. Young and W. H. Young.

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The authors discuss some of the contributions of W. H. and G. C. Young to real variable theory. They point out that most of the papers attributed to W. H. Young alone were in fact joint work with Grace.

Among the topics discussed are: inner limiting sets, now known as G_δ sets, and the classification of sets arising in analogous ways; questions about the derivatives of functions in a given Baire class; the unsolved problems of the characterisation of derivatives and whether the product of two derivatives is a derivative; and the Denjoy-Young-Saks theorem. The authors also describe later work by other authors. The Youngs' work on differentials and semicontinuous functions is mentioned.

The authors seem to have concentrated on the point-set aspects of real-variable theory; some other work in the field is omitted; for instance, the work on integration theory does not appear, nor does their joint expository paper of 1912 on variants of the Riesz-Fischer theorem.

Reviewed by *F. Smithies*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

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