MEASURES GENERATED BY A DIFFERENTIATION BASIS

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Theories which have been developed to carry out a program of differentiation for set functions normally proceed by imposing conditions on the differentiation basis similar to those encountered in Euclidean space. Thus, for example, a density theorem such as the classical Lebesgue result that for almost every x outside of the set A

$$\lim_{I \Rightarrow x} \frac{|I \cap A|}{|I|} = 0$$

as the intervals I shrink to x (see **[1]**), is commonly generalized to an abstract setting by replacing the intervals by a differentiation basis which retains the key Vitali property or some weaker version of it.

In this paper we show how a different viewpoint can be assumed: using an abstract differentiation basis, we construct certain auxiliary measures on the space and then state density theorems with regard to these measures. Should the basis possess appropriate Vitali properties it will turn out that the measures, so constructed, are equivalent to the original measure and that the density theorems are equivalent to standard ones in the literature.

We begin by listing the main definitions that we require:

(1) $\mathbf{F}$ is a differentiation basis on a set T provided that for every $t \in T$, $\mathbf{F}(t)$ is a filterbase of families of subsets of T. (*cf.* **[3**, p. 93]).

(2) If T is a topological space then it will moreover be assumed that $\mathbf{F}$ is finer than the topology in the sense that for every neighbourhood β of a point $t \in T$ there is an $F \in \mathbf{F}(t)$ with $\alpha \subset \beta$ for every $\alpha \in F$.

(3) If $\mathbf{F}$ is a differentiation basis on T then for any $A \subset T$ we write

$$\mathbf{F}[A] = \{\alpha : \alpha \in F \text{ for some } t \in A \text{ and } F \in \mathbf{F}(t)\}.$$

(4) A subset G of $\mathbf{F}[T]$ is said to be a *Vitali cover* of A if $G \cap F \neq \emptyset$ for every $t \in A$ and every $F \in \mathbf{F}(t)$.

(5) By an outer measure μ on T we shall mean a non-negative, extended real-valued function defined on all subsets of T with the properties that $\mu(\emptyset) = 0$ and $\mu(A) \leqslant \sum_{i=1}^{\infty} \mu(B_i)$ whenever $A \subset \bigcup_{i=1}^{\infty} \mu(B_i)$.

From an outer measure μ on a set T and a differentiation basis $\mathbf{F}$ on T we construct further outer measures denoted by μ_V and μ_λ $(1 \leqslant \lambda < +\infty)$. These constructions are motivated by the Vitali covering property and by a weaker version of that property introduced by Sion in **[2]**.

DEFINITION 1. *For any outer measure μ on T and any family G of subsets of T we define $V(\mu, G) = \sup\{\sum_{\alpha \in F} \mu(\alpha) : F \subset G\}$ where the supremum is with regard to all countable, disjoint subsets F of G (an empty supremum being taken as zero).*

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For any $A \subset T$ we define also

$$\mu_V(A) = \inf V(\mu, G)$$

where the infimum is with regard to all Vitali covers G of A relative to some fixed differentiation basis.

DEFINITION 2. *For any outer measure μ on T, any family G of subsets of T and any real number λ, $1 \leqslant \lambda < +\infty$, we define*

$$S_\lambda(\mu, G) = \sup \left\{ \sum_{\alpha \in F} \mu(\alpha) : F \subset G \right\}$$

where the supremum this time is with regard to all countable subsets F of G having the property that

$$\sum_{\alpha \in F} \mu(\alpha \cap \beta) \leqslant \lambda \mu(\beta)$$

for every $\beta \subset T$.

For any $A \subset T$ we define also

$$\mu_\lambda(A) = \inf S_\lambda(\mu, G)$$

where the infimum is again with regard to all Vitali covers G of A relative to some fixed differentiation basis.

THEOREM 1. *Let μ be an outer measure and $\mathbf{F}$ a differentiation basis on T. Then μ_V and μ_λ for $1 \leqslant \lambda < +\infty$ are outer measures on T and $\mu_V \leqslant \mu_\lambda$. If T is also a topological space, $A \subset T$, and $\mu(A) = \inf \{\mu(G) : G \text{ open}, G \supset A\}$ then $\mu_V(A) \leqslant \mu(A)$ and $\mu_\lambda(A) \leqslant \lambda \mu(A)$.*

Proof. To show that μ_V is an outer measure suppose that $A \subset \bigcup_{i=1}^{\infty} B_i$, that $\varepsilon > 0$, and that Vitali covers G_i of each B_i have been chosen so that

$$V(\mu, G_i) \leqslant \mu_V(B_i) + \varepsilon/2^i$$

for each i. Then $G = \bigcup_{i=1}^{\infty} G_i$ is a Vitali cover of A and so

$$\mu_V(A) \leqslant V(\mu, G) \leqslant \sum_{i=1}^{\infty} V(\mu, G_i) \leqslant \sum_{i=1}^{\infty} \mu_V(B_i) + \varepsilon.$$

As $\varepsilon > 0$ is arbitrary we have the required countable sub-additivity of μ_V; since $\mu_V(\varnothing) = 0$ is obvious it follows that μ_V is an outer measure. Similar arguments apply to each μ_λ.

The inequality $\mu_V \leqslant \mu_\lambda$ follows directly from the trivial inequality

$$V(\mu, G) \leqslant S_\lambda(\mu, G).$$

Finally the inequalities $\mu_V(A) \leqslant \mu(A)$ and $\mu_\lambda(A) \leqslant \lambda\mu(A)$ are easy consequences of the property (2) which relates the differentiation basis to the topology.

It can be verified that the outer measures μ and μ_V coincide in the presence of the Vitali covering property (V') of Sion [2, Definition 4.3] while the outer measures μ and μ_λ vanish together if the weaker covering property (V) of [2, Definition 3.7] holds. To illustrate the use to which these measures may be put we prove a density theorem which generalizes the classical Lebesgue density theorem. Note that our theorem and its corollary compare to Sion's [2] Theorems 4.2 and 4.7 respectively, and indeed reduce to those theorems when appropriate conditions are imposed on the differentiation basis.

The limits, $\lim_{W \Rightarrow x}$ and $\limsup_{W \Rightarrow x}$, are to be taken below in the sense of the filterbase $\mathbf{F}(x)$ and, as usual in these kind of theorems, we take $0/0 = 0$.

THEOREM 2. *Let μ be an outer measure and $\mathbf{F}$ a differentiation basis on a topological space T. Suppose that $A \subset T$, that $1 \leqslant \lambda < +\infty$, and that*

$$\inf \{\mu(A \setminus K) : K \subset A, K \text{ closed}\} = 0.$$

Then

$$\lim_{W \Rightarrow x} \frac{\mu(A \cap W)}{\mu(W)} = 0$$

for μ_λ-almost every x in $T \setminus A$.

Proof. For any $\varepsilon > 0$ and any natural number n choose a closed set $K \subset A$ so that $\mu(A \setminus K) < \varepsilon / n\lambda$. Let X_n denote the set

$$\left\{x \in T \setminus A : \limsup_{W \Rightarrow x} \mu(A \cap W)/\mu(W) > 1/n\right\}.$$

The proof is completed by showing that $\mu_\lambda(X_n) = 0$ for the set X,

$$X = \left\{x \in T \setminus A : \lim_{W \Rightarrow x} \mu(A \cap W)/\mu(W) \neq 0\right\},$$

is the union of the X_n ($n = 1, 2, \ldots$) and so is of μ_λ-measure zero as required.

Let G denote the family of all sets $\alpha \in \mathbf{F}[X_n]$ with $\alpha \subset T \setminus K$ and

$$\mu(\alpha) \leqslant n\mu(\alpha \cap A).$$

It is straightforward to verify that G is a Vitali cover of X_n and therefore that

$$\mu_\lambda(X_n) \leqslant S_\lambda(\mu, G).$$

Now if F is any countable subset of G with the property that

$$\sum_{\alpha \in F} \mu(\alpha \cap \beta) \leqslant \lambda\, \mu(\beta)$$

for all $\beta \subset T$, then we must have

$$\begin{aligned} \sum_{\alpha \in F} \mu(\alpha) &\leqslant n \sum_{\alpha \in F} \mu(\alpha \cap A) \\ &= n \sum_{\alpha \in F} \mu(\alpha \cap (A \setminus K)) \\ &\leqslant n\,\lambda\,\mu(A \setminus K) \\ &< \varepsilon \end{aligned}$$

and hence that $\mu_\lambda(X_n) \leqslant S_\lambda(\mu, G) < \varepsilon$. As $\varepsilon > 0$ is arbitrary it follows that each set X_n is of μ_λ-measure zero which completes the proof of the theorem.

COROLLARY. *Let μ and ν be outer measures and $\mathbf{F}$ a differentiation basis on a* *topological space T. Suppose that* $A \subset T$ *and that*

$$\inf\{\mu(A \setminus K) : K \subset A,\ K \text{ closed}\} = 0$$

and

$$\inf\{\nu(A \setminus K) : K \subset A,\ K \text{ closed}\} = 0.$$

Then $\lim_{W \Rightarrow x} \nu(A \cap W)/\mu(W) = 0$ *for* μ_V*-almost every* x *in* $T \setminus A$.

Proof. Set $\phi(\beta) = \nu(\beta \cap A)$ for all $\beta \subset T$ and apply Theorem 2 to the outer measure $\mu+\phi$. Then we have

$$\lim_{W \Rightarrow x} \frac{(\mu+\phi)(A \cap W)}{(\mu+\phi)(W)} = 0$$

almost everywhere in $T \setminus A$ for any outer measure $(\mu+\phi)_\lambda$; but

$$(\mu+\phi)_\lambda \geqslant (\mu+\phi)_V \geqslant \mu_V$$

so that this limit is true, *a fortiori*, μ_V-almost everywhere in $T \setminus A$. Also by the theorem we have

$$\lim_{W \Rightarrow x} \frac{\mu(A \cap W)}{(\mu+\phi)(W)} \leqslant \lim_{W \Rightarrow x} \frac{\mu(A \cap W)}{\mu(W)} = 0$$

μ_V-almost everywhere in $T \setminus A$.

Putting these together shows that

$$\lim_{W \Rightarrow x} \frac{\phi(A \cap W)}{(\mu+\phi)(W)} = \lim_{W \Rightarrow x} \frac{\phi(W)}{\mu(W)+\phi(W)} = 0$$

and hence also that

$$\lim_{W \Rightarrow x} \frac{\phi(W)}{\mu(W)} = 0$$

μ_V-almost everywhere in $T \setminus A$ as required.

References

1. S. Saks, *Theory of the integral* (Warsaw, 1937).
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