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OBITUARY

Ralph Henstock, 1923–2007

Integration theorist Ralph Henstock died on 17 January 2007 in Coleraine, Northern Ireland, after a short illness.

1. *Early life and education*

Ralph Henstock ⟨**1**⟩ was born in the coal-mining village of Newstead, near Nottingham, UK, on 2 June 1923; the only child of mineworker and former coalminer William Henstock and Mary Ellen Henstock (née Bancroft). On the Henstock side, he was descended from seventeenth-century Flemish immigrants called Hemstok.

Because of his early academic promise, it was expected that Henstock would attend Nottingham University, where his father and uncle had received technical education, but as it turned out he won scholarships that enabled him to study mathematics at St. John's College, Cambridge. He studied there from October 1941 until November 1943, when he was sent for war service to the Ministry of Supply's department of Statistical Method and Quality Control in London.

This work did not satisfy him, and so he enrolled at Birkbeck College, London, where he joined the weekly seminar of Professor Paul Dienes, which was then a focus for mathematical activity in London. Henstock wanted to study divergent series, but Dienes prevailed upon him to get involved in the theory of integration, thereby setting him on course for his life's work.

He was awarded the Cambridge B.A. in 1944 and began research for a Ph.D. in London, which he gained in December 1948 with a thesis entitled *Interval Functions and their Integrals*, an extension of J. C. Burkill's theory. His Ph.D. examiners were J. C. Burkill and H. Kestelman.

In 1947, he returned briefly to Cambridge to complete the undergraduate mathematical studies that had been truncated by his Ministry of Supply work.

2. *The Henstock integral*

Henstock was a distinguished analyst who specialized in the theory of integration. From initial studies of the Burkill and Ward integrals, he formulated an integration process whereby the domain of integration is suitably partitioned for Riemann sums to approximate the integral of a function. His methods led to an integral on the real line that was very similar in construction and simplicity to the Riemann integral, but which included the Lebesgue integral and, in addition, allowed non-absolute convergence.

The difference between absolute convergence and non-absolute convergence is illustrated by the familiar series $\sum_{j=1}^{\infty} a_j$, with $a_j = (-1)^j j^{-1}$. This is conditionally or non-absolutely convergent. Since $\sum_{j=1}^{\infty} |a_j|$ diverges, the series is not absolutely convergent.

Now consider the function

$$f(x) = \begin{cases} 2x \sin x^{-2} - 2x^{-1} \cos x^{-2} & \text{if } 0 < x \leqslant 1, \\ 0 & \text{if } x = 0. \end{cases}$$

This is the derivative of the function

$$F(x) = \begin{cases} x^2 \sin x^{-2} & \text{if } 0 < x \leqslant 1, \\ 0 & \text{if } x = 0. \end{cases}$$

In other words, $F(x)$ is the 'indefinite integral' of $f(x)$. Therefore, for any $\alpha > 0$, we expect that

$$\int_0^{\alpha} f(x)\, dx = F(\alpha) - F(0) = \alpha^2 \sin \alpha^{-2}.$$

However, $f(x)$ is not integrable in the senses of Riemann or Lebesgue. The problem here is analogous to the failure of $\sum_{j=1}^{\infty} (-1)^j j^{-1}$ to converge absolutely. The integrals of Perron and Denjoy ⟨**2**⟩ provide definitions of the integral that admit non-absolute convergence in this case.

However, a simpler resolution of problems such as this is given by the Henstock integral as follows: *the function $f(x)$ is integrable in $[0, \alpha]$ with integral $\beta = \int_0^{\alpha}$ if, given $\varepsilon > 0$, for each $x \in [0, \alpha]$ we can find a function $\delta(x) > 0$ defined on $[0, \alpha]$ so that*

$$\left| \sum_{j=1}^{n} f(x_j)(u_j - u_{j-1}) - \beta \right| < \varepsilon$$

for every partition $0 = u_0 < u_1 < \ldots < u_n = \alpha$ of $[0, \alpha]$, satisfying

$$x_j - \delta(x_j) < u_{j-1} \leqslant x_j \leqslant u_j < x_j + \delta(x_j). \tag{1}$$

This solution to the problem of non-absolute integration is particularly elegant because, when the variable $\delta(x) > 0$ is replaced by the constant $\delta > 0$, the definition reduces to the familiar Riemann integral.

The integral $\int_{-\infty}^{\infty} \exp \iota x^2\, dx$, where $\iota = \sqrt{-1}$, is a basic element of the Feynman path integral formulation of quantum mechanics and quantum field theory. Unlike the function $f(x)$ above, the function $g(x) = \exp \iota x^2$ is continuous in its domain of integration. It is not absolutely integrable in the Lebesgue sense but, like $f(x)$, the function $g(x)$ is non-absolutely integrable in the Henstock sense, with integral (known as Fresnel's integral) equal to $\sqrt{\iota\pi}$.

To get a sense of how the Henstock Riemann sums converge for this function, for each x consider the zero z of $g(x)$ that is closest to x, but not equal to x, and thus $|x - z| > 0$. Then choose $\delta(x) < |x - z|$. Since the cycles of $g(x) = \cos x^2 + \iota \sin x^2$ are alternately positive and negative, we can go on to ensure that terms of the Riemann sum for $g(x)$ can be grouped so that the groups of terms are alternately positive- and negative-valued, with successively smaller absolute values tending to zero, analogously to the non-absolutely convergent series $\sum_{j=1}^{\infty}(-1)^j j^{-1}$.

These ideas on a non-absolute version of Riemann integration were developed by Henstock from the mid-1950s. Independently, Kurzweil ⟨**5**⟩ developed a similar Riemann-type integral on the real line. The resulting integral is now known as the Henstock–Kurzweil integral. On the real line, it is equivalent to the Denjoy–Perron integral, but has a much simpler definition and is generally much easier to work with. An absolutely convergent version of this integral, equivalent to the Lebesgue integral on the real line, was developed by McShane ⟨**6**⟩. In McShane's development, instead of the condition (1), we have

$$x_j - \delta(x_j) < u_{j-1} < u_j < x_j + \delta(x_j);$$

in other words, x_j may be outside of the interval $[u_{j-1}, u_j]$.

In the following decades, Henstock developed extensively the distinctive features of his theory, inventing the concepts of division spaces or integration bases to demonstrate in general settings the essential properties and characteristics of mathematical integration in all its forms. His theory provides a unified approach to many problems that were considered earlier by different methods using different types of non-absolute integrals [**43**]. Now many of them can be solved using different kinds of Henstock integral, just choosing an appropriate integration basis (or division space in Henstock's own terminology).

The theory of integration (including measure) is the basis for the study of probability and random variation. Thus Henstock's Riemann-type integration theory has relevance to our understanding of random variation. Henstock addressed this issue in many of his published works, in which he gave interpretations of probability, of the statistical analysis of data, and of random processes. His analysis of Feynman's non-absolute integrals in quantum mechanics brings this subject properly into the domain of random variation; see ⟨**7**⟩.

3. *Review of Henstock's work*

In a personal report ⟨**3**⟩ written in 1984, Henstock described his research interests as follows.

> *My research is in the following pure mathematical fields*:
>
> (1) *summability of series and integrals,*
> (2) *integration theory,*
>
> *and especially in problems that link the two. In* (1) *an essential tool is often the Banach–Steinhaus theorem of functional analysis, with Sargent's modification. For example, if* $\int_a^b f\,dg$ *exists for every Baire- or Borel-measurable function, to prove that* g *is of bounded variation on* $[a, b]$. *Putting a summability factor into the definition of the integral leads to a generalization of Burkill's Cesàro–Perron integrals and the Marcinkiewicz–Zygmund integral. These are of Perron type, defined by inequalities of the type*
>
> $$\int_x^{x+h} \{F(t) - F(x)\}\, d_t N(x, h; t) \geqslant \int_x^{x+h} f(x)(t - x)\, d_t N(x, h; t),$$
>
> *with similar inequalities for* $[x - h, x]$, *and the problem is to find the necessary conditions on* N. *Out of this came the variational integral, which then led to the Riemann-complete or generalized Riemann integral, the so-called Kurzweil–Henstock integral. This integral includes the Riemann, Riemann–Stieltjes, Lebesgue, Radon, Denjoy special, and Perron integrals, using Riemann's*

original sums but a different limit. Set-valued functions, the integrals of which have applications in economics. Wiener-type integration has applications to various stochastic processes such as white noise. Feynman-type integration has applications in quantum theory. An integral that includes the Paley–Wiener–Zygmund integral has applications for stochastic integration, as does an integral equivalent to the Itô integral. The most general form of the generalized Riemann integral can be used to define all these integrals except those defined by the functions N.

The theory of integration with which Henstock is associated arose from his study of the problem of summability described above, but did not completely resolve this problem to his satisfaction. Therefore, in a sense, his successful mathematical accomplishments are a by-product of a different project that he felt was incomplete. Such things happen in a life of high achievement.

Another ancillary problem, which he addressed at various times in the course of his life, was the mathematical analysis of random variation. He first encountered this subject in a very practical way when, in November 1943, he was withdrawn temporarily from his mathematical studies at St. John's College, Cambridge, and assigned to the British Ministry of Supply to do statistical work.

His experience as a civil servant generated in him a visceral dislike of working for the Government; but he retained a life-long interest in the analysis of random data. Therefore he took a course of study in stochastic theory from M. S. Bartlett in 1947, and in 1958 he became a Fellow of the Royal Statistical Society.

Henstock's approach to the theory of integration builds on the nineteenth-century theory of Riemann, and is conceptually different from and independent of the early twentieth-century integration theory of Lebesgue. Thus one can imagine that the mathematical theory of probability founded on the work of Kolmogorov (as described in, for instance, *Foundations of the theory of probability* ⟨**4**⟩) could conceivably have been based on a Riemann-type integration rather than the Lebesgue approach used by Kolmogorov and his successors in probability theory.

Henstock's writings give a strong sense of how such an alternative development of probability theory should be accomplished. His 1963 book (*Theory of integration* [**18**]) includes a chapter on probability, as does his *Lectures on the theory of integration* [**38**]. Here, Henstock reviews three different interpretations, including Kolmogorov's, of the probability concept. Placing emphasis on the role of actual statistical data, he discusses the classification or partitioning of numerical data into disjoint real intervals $\{I\}$, which is often the first practical step in the numerical analysis of such data. He provides two ways to define the probability that a numerical measurement x takes a value in a set X. Accordingly, $\mathrm{Prob}(X)$ can be taken to be

$$\mathrm{Prob}(X) = \int_{-\infty}^{\infty} \mathbf{1}_X(x)\, dP = \int_X dP \tag{2}$$

or

$$\mathrm{Prob}(X) = V(P;\ X). \tag{3}$$

If we take (3) (the P-variation of the set X; see ⟨**7**, p. 26⟩) as the definition of probability, then every set X has a mathematical probability. If we take (2) (the Henstock integral of P in the set X) as the definition, then, as in the Kolmogorov theory, only certain sets X have a probability measure (and, for those sets, (3) gives the same value). In that case, if the function P is a probability measure in the sense of Kolmogorov, then Henstock's approach gives exactly the same measurable sets as Kolmogorov's.

The underlying simplicity of the Henstock–Kurzweil integral has reinvigorated the subject of mathematical integration and the theory now has many practitioners and exponents. It has proved useful in differential and integral equations, harmonic analysis, probability theory, and quantum mechanics, where the random variables of Feynman integration are not absolutely integrable and therefore are not amenable to the methods of Lebesgue integration and classical probability theory. Numerous monographs and texts have appeared since 1980 and there have been several conferences devoted to the theory; but the simplicity of the underlying concept can sometimes give rise to naive expectations of the subject, which is, in reality, deep and subtle.

Initially a research specialism, it is nowadays increasingly taught in standard courses in mathematical analysis.

4. *Career and publications*

Henstock was the author of 46 journal papers during the period 1946–2006. He published four books on analysis (*Theory of integration* [**18**]; *Linear analysis* [**24**]; *Lectures on the theory of integration* [**38**]; *The general theory of integration* [**43**]). He wrote 171 reviews for MathSciNet. In 1994, he was awarded the Andy Prize of the XVIII Summer Symposium in Real Analysis. His academic career comprised the following stages: he began as Assistant Lecturer, Bedford College for Women, 1947–48; then Assistant Lecturer at Birkbeck, 1948–51; Lecturer, Queen's University Belfast, 1951–56; Lecturer, Bristol University, 1956–60; Senior Lecturer and Reader, Queen's University Belfast, 1960–64; Reader, Lancaster University, 1964–70; Chair of Pure Mathematics, New University of Ulster, 1970–88; and Leverhulme Fellow 1988–91.

Henstock married Marjorie Jardine in 1949 and is survived by their son John. A devoted Methodist, he had an abiding interest in poetry, and the lasting impression he made was one of gentle sincerity, kindness, and amiability. The integrity and conscientiousness he displayed in his scientific work were mirrored in his generous relationships with colleagues, collaborators, and students. As a mathematician and as a man, his loss is deeply felt by all of them.

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