SPACES OF CONDITIONALLY INTEGRABLE FUNCTIONS

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1. *Introduction*

Let CL(a, b), D*(a, b) and D(a, b) denote the linear spaces of functions integrable on the interval $[a, b]$ in the Cauchy–Lebesgue, Denjoy–Perron and Denjoy–Khintchine senses respectively and topologized by the seminorm

$$f \to \sup \left\{ \left| \int_a^t f(\xi) d\xi \right| ; \; a \leqslant t \leqslant b \right\}$$

where the integral is in the appropriate sense; let **CL**(a, b), **D***(a, b) and **D**(a, b) be the associated separated spaces. Miss W. L. C. Sargent [3] has established several theorems of Banach–Steinhaus type for these spaces; from a modern point of view it is evident that these results may be interpreted as proving that each of the above spaces is barrelled. In this paper we present a direct proof of this observation.

In particular these spaces provide the earliest examples of barrelled spaces which are meager and normable; the standard example of such a space [1; p.157, Ex. 10] is perhaps less natural and less accessible as it is obtained essentially from the fact that the tensor product of two Banach spaces equipped with its projective tensor product topology is barrelled and normable but not, in general, a Baire space.

2. *The main theorems*

Unless otherwise specified all intervals will be open subintervals of a fixed interval (a, b) and will be denoted by the letters I and J. A real- (or complex-) valued function F whose domain is the collection of all subintervals of (a, b) and which has the property $F(\alpha, \gamma) = F(\alpha, \beta) + F(\beta, \gamma)$ for all $a \leqslant \alpha < \beta < \gamma \leqslant b$ is said to be an additive interval function on (a, b). If $J \subset (a, b)$ then $F(J \cap (:))$ denotes the additive interval function $I \to F(J \cap I)$ on (a, b) where, for consistency, $F(\emptyset) = 0$.

By **C**(a, b) we mean the Banach space of all *continuous* additive interval functions on (a, b), with the norm $F \to \|F\| = \sup\{|F(J)|; J \subset (a, b)\}$. If **X** is a subspace of **C**(a, b) and $J \subset (a, b)$ we define the subspace $\mathbf{X}(J) = \{F(J \cap (:)); F \in \mathbf{X}\}$.

Let **X** be an arbitrary locally convex separated topological vector space and suppose B is a barrel in **X**: we shall require the two following fundamental properties of barrels.

[2.1] B is either nowhere dense in **X** or else B is a neighbourhood of the origin. (See [**1**, proof of Prop. 1, pp. 1–2])

[2.2] B absorbs every complete bounded absolutely convex set in **X**. (See [**1**, Lemma 1, p. 21]).

We may now state our principal results.

THEOREM 1. *Let* **X** *be a subspace of* **C**(a, b) *with the properties*

C1. *If* $F \in \mathbf{X}$ *and* $J \subset (a, b)$ *then* $F(J \cap (:)) \in \mathbf{X}$.

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C2. *If* $a \leqslant c \leqslant b$, $F \in \mathbf{C}(a, b)$ and $F(J \cap (:)) \in \mathbf{X}$

for every interval J *with* $c \notin \bar{J}$ *then* $F \in \mathbf{X}$. *Then* $\mathbf{X}$ *is barrelled.*

THEOREM 2. (Sargent). *Each of the spaces* $\mathbf{CL}(a, b)$, $\mathbf{D}^*(a, b)$ *and* $\mathbf{D}(a, b)$ *is barrelled.*

Proof. Each of the above spaces is evidently topologically isomorphic to a subspace of $\mathbf{C}(a, b)$ which satisfies the two (Cauchy) properties C1 and C2 of Theorem 1.

Proof of Theorem 1. Let B be a barrel in $\mathbf{X}$ and suppose, contrary to the theorem, that B is nowhere dense in $\mathbf{X}$ (*cf.* [2.1]). For every interval $I \subset (a, b)$ write $B(I) = B \cap \mathbf{X}(I)$ so that $B(I)$ is clearly a barrel in $\mathbf{X}(I)$. Let $I = (\alpha, \gamma)$, $I_1 = (\alpha, \beta)$ and $I_2 = (\beta, \gamma)$ be subintervals of (a, b).

[2.3] If $B(I)$ is nowhere dense in $\mathbf{X}(I)$ then either $B(I_1)$ is nowhere dense in $\mathbf{X}(I_1)$ or $B(I_2)$ is nowhere dense in $\mathbf{X}(I_2)$.

To prove this observe that $B(I_1) + B(I_2) \subset 2B(I)$ and that $\mathbf{X}(I)$ is the topological direct sum of the subspaces $\mathbf{X}(I_1)$ and $\mathbf{X}(I_2)$. Thus if $B(I_1)$ and $B(I_2)$ are neighbourhoods of the origin in $\mathbf{X}(I_1)$ and $X(I_2)$, it follows that $B(I)$ is a neighbourhood of the origin in $\mathbf{X}(I)$, which proves [2.3].

Now if B is nowhere dense in $\mathbf{X}$, we can find by repeated subdivision of (a, b), using [2.3], a decreasing sequence of intervals $\{I_n\}$ such that $\bigcap_{n=1}^{\infty} \bar{I}_n$ contains a single point $c \in [a, b]$ and such that each $B(I_n)$ is nowhere dense in $\mathbf{X}(I_n)$.

A further application of [2.3] proves the existence of a set $E \subset [a, b]$ such that either

(i) $E \subset [a, c)$ and c is a limit point of E so that $B(t, c)$ is nowhere dense in $\mathbf{X}(t, c)$ for all $t \in E$, or

(ii) $E \subset (c, b]$ and c is a limit point of E so that $B(c, t)$ is nowhere dense in $\mathbf{X}(c, t)$ for all $t \in E$.

We need only consider the case (i) as symmetric arguments apply to (ii): choose $\varepsilon_1 \in E$ and, since $B(\varepsilon_1, c)$ is nowhere dense in $\mathbf{X}(\varepsilon_1, c)$, choose $G_1 \in \mathbf{X}(\varepsilon_1, c)$ with $G_1 \notin B$ and $\|G_1\| < \frac{1}{2}$. But B is closed and $\lim_{\varepsilon \to c-} G_1((\varepsilon_1, \varepsilon) \cap (:)) = G_1$ in $\mathbf{X}$ so that there exists $\varepsilon_2 \in E$ with $\varepsilon_2 > \varepsilon_1$ such that, if $F_1 = G_1((\varepsilon_1, \varepsilon_2) \cap (:))$, we have $F_1 \in \mathbf{X}(\varepsilon_1, \varepsilon_2)$, $F_1 \notin B$ and $\|F_1\| < \frac{1}{2}$. Repeating these arguments we find, inductively, a sequence of points $\{\varepsilon_n\}$ in E with $\varepsilon_1 < \varepsilon_2 < \ldots$ and $\lim \varepsilon_n = c$ and a sequence $\{F_n\}$ such that $F_n \in \mathbf{X}(\varepsilon_n, \varepsilon_{n+1})$, $F_n \notin nB$ and $\|F_n\| < 1/2^n$.

Let A be the absolutely convex envelope of the set $\{F_1, F_2, F_3, \ldots\}$ and let $\bar{A}$ be the closure of A in the space $\mathbf{C}(a, b)$: every element F of $\bar{A}$ is of the form $F = \sum_{k=1}^{\infty} \lambda_k F_k$ for some sequence of scalars $\{\lambda_k\}$ with $\sum_{k=1}^{\infty} |\lambda_k| \leqslant 1$. (Note that each such series obviously converges in $\mathbf{C}(a, b)$ and also that $\|F\| \leqslant 1$ for all $F \in \bar{A}$). Let now $\varepsilon_1 < \varepsilon < c$ and observe that

$$F((\varepsilon_1, \varepsilon) \cap (:)) = \sum_{k=1}^{\infty} \lambda_k F_k\,((\varepsilon_1, \varepsilon) \cap (:))$$

where only a finite number of terms are nonzero; thus for all $\varepsilon_1 < \varepsilon < c$

$$F((\varepsilon_1, \varepsilon) \cap (:)) \in \mathbf{X}$$

and by C2 it follows that $F \in \mathbf{X}$.

Hence $\bar{A} \subset \mathbf{X}$ and in particular we have $\bar{A}$ is a complete, bounded, absolutely convex set in $\mathbf{X}$; by [2.2] then B absorbs $\bar{A}$ which is impossible since, by construction B does not even absorb the set $\{F_1, F_2, F_3, \ldots\} \subset \bar{A}$ and this contradiction establishes the theorem.

3. *Remarks on Sargent's β-spaces*

We conclude by relating a concept introduced by W. L. C. Sargent to the modern concept of a barrelled space; for a more detailed discussion of similar ideas in the broader context of topological groups see [**2**].

A topological vector space $\mathbf{X}$ is said to be a β-space (*cf.* [**2**; p. 314; **3**; p. 440]) if there is no sequence of sets $\{Q_n\}$ such that

S1. $0 \in Q_1$, $Q_n - Q_n \subset Q_{n+1}$ and $\bigcup_{n=1}^{\infty} Q_n = \mathbf{X}$, and

S2. Every set Q_n is nowhere dense in $\mathbf{X}$.

THEOREM 3. *Every locally convex β-space is barrelled, but not conversely; every Baire space is a β-space, but not conversely.*

Proof. Let $\mathbf{X}$ be a locally convex β-space and let B be a barrel in $\mathbf{X}$: the sequence $Q_n = 2^n B$, $n = 1, 2, \ldots$, evidently satisfies S1 and so, since $\mathbf{X}$ is a β-space, one of the sets Q_n is somewhere dense. Thus, by [2.1], B is a neighbourhood of the origin and $\mathbf{X}$ is barrelled.

The converse is false: let $\mathbf{X}$ be a linear space with a denumerable (Hamel) basis $\{x_1, x_2, x_3, \ldots\}$ equip $\mathbf{X}$ with its finest locally convex topology and let Q_n be the linear span of the set $\{x_1, x_2, x_3, \ldots, x_n\}$. Then $\{Q_n\}$ satisfies S1 and S2 so that $\mathbf{X}$ is not a β-space; however, it is well-known that $\mathbf{X}$ is barrelled. (In fact the example [**1**, p. 157, ex. 10] even exhibits a space which is normable and barrelled but not a β-space.) Finally it is obvious that a Baire space is a β-space and [**3**] provides examples of meager β-spaces.

References

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