COVERING SYSTEMS AND DERIVATES IN HENSTOCK DIVISION SPACES

B. S. THOMSON

The theory of division spaces introduced by Henstock in [2, 3] in order to simplify and unify certain areas in the theory of integration is particularly well suited to accomodating various parts of the theory of covering systems and differentiation. In this paper we present an introduction to these ideas in a quite general setting. Our approach involves the elegant idea due to Henstock in [1] of introducing a further measure on the space, called the *inner variation*; proving various results on derivates which hold almost everywhere with respect to this measure; and then imposing " Vitali " conditions to assure that the measure coincides with the original one.

1. *Division spaces, variation*

Let T be a set and $\mathbf{I}$ a collection of pairs (I, x) $(I \subseteq T, x \in T)$. A finite subset $\mathbf{D}$ of $\mathbf{I}$ is said to be a *division* if the sets in $\{I : (I, x) \in \mathbf{D}\}$ are disjoint. For a division $\mathbf{D}$ we write $\sigma(\mathbf{D}) = \bigcup\{I : (I, x) \in \mathbf{D}\}$ and we call any set $E = \sigma(\mathbf{D})$ *an elementary set* and $\mathbf{D}$ *a division of* E.

If $X \subseteq T$ and $\mathbf{S} \subseteq \mathbf{I}$ we define

$$\mathbf{S}(X) = \{(I, x) \in \mathbf{S}: \; I \subseteq X\}, \tag{1.1}$$

$$\mathbf{S}[X] = \{(I, x) \in \mathbf{S}: \; x \in X\}. \tag{1.2}$$

Definition 1. The ordered triple $(T, \mathfrak{A}, \mathbf{I})$ is said to be a division system provided $\mathfrak{A}$ is a family of subsets of $\mathbf{I}$ such that

(i) If $x \in T$ and $\mathbf{S} \in \mathfrak{A}$ then $(\emptyset, x) \in \mathbf{S}$.

(ii) $\mathfrak{A}$ is directed downwards by set inclusion.

Condition (i) is unnecessary but makes the covering system definitions of the next section easier to apply.

A division system $(T, \mathfrak{A}, \mathbf{I})$ is said to be *fully decomposable* (resp. *decomposable*) if to every family (resp. countable family) $\{X_i : i \in I\}$ of disjoint subsets of T and every $\{\mathbf{S}_i : i \in I\} \subseteq \mathfrak{A}$ there exists an $\mathbf{S} \in \mathfrak{A}$ with $\mathbf{S}[X_i] \subseteq \mathbf{S}_i[X_i]$ for all $i \in I$.

If μ is a real-valued function defined on $\mathbf{I}$ we define the *variation* of μ with respect to an $\mathbf{S} \subseteq \mathbf{I}$ as

$$V(\mu, \mathbf{S}) = \sup\,(\mathbf{D}) \sum |\mu(I, x)|, \tag{1.3}$$

where the supremum is taken over all divisions $\mathbf{D}$ $(\mathbf{D} \subseteq \mathbf{S})$ and $(\mathbf{D})\sum$ denotes summation over all $(I, x) \in \mathbf{D}$, an empty sum by convention being zero.

If $\mathfrak{A}$ is a family of subsets of $\mathbf{I}$ then we define also

$$V(\mu, \mathfrak{A}) = \inf\{V(\mu, \mathbf{S}): \; \mathbf{S} \in \mathfrak{A}\}. \tag{1.4}$$

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If $(T, \mathfrak{A}, \mathbf{I})$ is a division system and μ a real-valued function on $\mathbf{I}$ we write

$$\mu^*(X) = V(\mu, \mathfrak{A}[X]) \quad (X \subseteq T),$$

where $\mathfrak{A}[X] = \{\mathbf{S}[X]: \ \mathbf{S} \in \mathfrak{A}\}$. A fundamental result is that

(1.5) *If $(T, \mathfrak{A}, \mathbf{I})$ is a decomposable division system then μ^* is an (outer) measure on T.* (*cf.* [**2**, **3**]).

Let $\mathbf{E}$ be the collection of all elementary sets for a division system $(T, \mathfrak{A}, \mathbf{I})$: a real-valued function F on $\mathbf{E}$ is said to be *additive* if $F(E) = (\mathbf{D}) \sum F(I)$ for every division $\mathbf{D}$ of E and every $E \in \mathbf{E}$. If f is a real-valued function on T and μ a real-valued function on $\mathbf{I}$ we say $F = \int f d\mu$ exists (relative to $(T, \mathfrak{A}, \mathbf{I})$) provided $V(\hat{F} - f\mu, \mathfrak{A}) = 0$ for some additive function F (which need not be unique) where we write $\hat{F}$ as the function $(I, x) \to F(I)$ and $f\mu$ as the function $(I, x) \to f(x)\, \mu(I, x)$.

2. *Covering systems, Vitali systems*

Let $(T, \mathfrak{A}. \mathbf{I})$ be an arbitrary division system. A subset $\mathbf{N}$ of $\mathbf{I}$ is said to *cover a set* $X \subseteq T$ *finely* if $\mathbf{N} \cap \mathbf{S}[\{x\}] \neq \emptyset$ for every $x \in X$ and for every $\mathbf{S} \in \mathfrak{A}$. If $\mathbf{N}$ covers T itself finely then we call $\mathbf{N}$ a *covering system* on $(T, \mathfrak{A}, \mathbf{I})$ while if $\mathbf{F} \subseteq \mathbf{N}[X]$ covers X finely we say $\mathbf{F}$ is an $\mathbf{N}$-*cover for* X.

Suppose μ is a real-valued function on $\mathbf{I}$ and $\mathbf{N}$ is a covering system on the division system $(T, \mathfrak{A}, \mathbf{I})$. Associated with μ and $\mathbf{N}$ will be a measure on T called the $(\mu, \mathbf{N})$-inner variation defined as follows:

$$IV_{\mathbf{N}}(\mu, X) = \inf V(\mu, \mathbf{F}),$$

where the infimum is with regard to all $\mathbf{F} \subseteq \mathbf{N}[X]$ that are $\mathbf{N}$-covers of X. In terms of (1.4) then $IV_{\mathbf{N}}(\mu, X) = V(\mu, \mathfrak{B})$ where $\mathfrak{B}$ is the family of all $\mathbf{F}$ which are $\mathbf{N}$-covers of X. (Note that this definition does *not* coincide with that of Henstock in [**1**].)

We have the following elementary properties for an arbitrary covering system on $(T, \mathfrak{A}, \mathbf{I})$ with μ and μ' real functions on $\mathbf{I}$.

(2.1) $$0 \leqq IV_{\mathbf{N}}(\mu, X) \leqq +\infty, \quad IV_{\mathbf{N}}(\mu, \emptyset) = 0.$$

(2.2) $$IV_{\mathbf{N}}(\mu, X) \leqq \mu^*(X).$$

(2.3) $$IV_{\mathbf{N}}(\mu + \mu', X) \leqq IV_{\mathbf{N}}(\mu, X) + V(\mu', \mathfrak{A}[X]).$$

(2.4) $$\text{If } X = \textstyle\bigcup_{k=1}^{\infty} X_k \text{ then } IV_{\mathbf{N}}(\mu, X) \leqq \sum_{k=1}^{\infty} IV_{\mathbf{N}}(\mu, X_k).$$

Properties (2.1) and (2.4) show that the function $X \to IV_{\mathbf{N}}(\mu, X)$ is always a measure on T. We prove only the last of the relations as the others are straightforward.

For each integer k let $\mathbf{F}_k$ be an $\mathbf{N}$-cover of X_k chosen so that

$$V(\mu, \mathbf{F}_k) \leqq IV_{\mathbf{N}}(\mu, X_k) + \varepsilon/2^k,$$

and let $\mathbf{F} = \bigcup_{k=1}^{\infty} \mathbf{F}_k$. Clearly $\mathbf{F}$ is an $\mathbf{N}$-cover of X so that

$$IV_{\mathbf{N}}(\mu, X) \leqq V(\mu, \mathbf{F}) \leqq \sum_{k=1}^{\infty} V(\mu, \mathbf{F}_k) \leqq \sum_{k=1}^{\infty} IV_{\mathbf{N}}(\mu, X_k) + \varepsilon$$

and letting $\varepsilon \to 0$ proves (2.4).

The most interesting applications of the theory occur when the $(\mu, \mathbf{N})$-inner variation coincides with the usual variation μ^*. We formalize this in a definition.

Definition 2. A covering system $\mathbf{N}$ on a division system $(T, \mathfrak{A}, \mathbf{I})$ is said to be a *Vitali system* for the function μ if for every $X \subseteq T$

$$IV_{\mathbf{N}}(\mu, X) = \mu^*(X).$$

The motivation for the term lies in the following theorem relating our definition to the more standard one (*cf.* Saks [**4**; p. 109]).

THEOREM 2.5. *Let* $\mathbf{N}$ *be a covering system on a decomposable division system* $(T, \mathfrak{A}, \mathbf{I})$ *and suppose that*

(2.5.1) $\mu^*(I) \leqq |\mu(I, x)|$ *for every* $(I, x) \in \mathbf{N}$,

(2.5.2) *for every* $X \subseteq T$ *and every* $\mathbf{N}$*-cover* $\mathbf{F}$ *of* X *there is a sequence* (I_k, x_k) $(k = 1, 2, \ldots)$ *contained in* $\mathbf{F}$ *with the* $\{I_k\}$ *disjoint such that*

$$\mu^*\left(X \setminus \bigcup_{k=1}^{\infty} I_k\right) = 0.$$

Then $\mathbf{N}$ *is a Vitali system for* μ *on* $(T, \mathfrak{A}, \mathbf{I})$.

Proof. Let $X \subseteq T$ and let $\mathbf{F}$ be an arbitrary $\mathbf{N}$-cover of X. If (I_k, x_k) $(k = 1, 2, \ldots)$ is the sequence contained in $\mathbf{F}$ having the properties (2.5.2) then using (2.5.1) and (1.5)

$$\begin{aligned} \mu^*(X) &\leqq \mu^*\left(X \setminus \bigcup_{k=1}^{\infty} I_k\right) + \sum_{k=1}^{\infty} \mu^*(X \cap I_k) \\ &\leqq \sum_{k=1}^{\infty} \mu^*(X \cap I_k) \\ &\leqq \sum_{k=1}^{\infty} \mu^*(I_k) \leqq \sum_{k=1}^{\infty} |\mu(I_k, x_k)| \leqq V(\mu, \mathbf{F}). \end{aligned}$$

From this it follows that $\mu^*(X) \leqq IV_{\mathbf{N}}(\mu, X)$ which with (2.2) completes the proof.

3. *Derivates*

Throughout this section $\mathbf{N}$ will be a covering system on a division system $(T, \mathfrak{A}, \mathbf{I})$ and μ and ψ will be real-valued functions defined on $\mathbf{I}$. No other assumptions except those stated are necessary for our results: however if $\mathbf{N}$ is a Vitali system for μ then the $(\mu, \mathbf{N})$-inner variation may be replaced by the variation μ^* wherever it occurs.

Definition 3. The *upper* and *lower derivates* of ψ with respect to μ relative to $\mathbf{N}$ at a point x are defined as

(3.1) $$\bar{D}_{\mathbf{N}}(\psi \mid \mu : x) = \inf_{\mathbf{S} \in \mathfrak{A}} \sup \{\psi(I, x)/\mu(I, x) : (I, x) \in \mathbf{S} \cap \mathbf{N}\},$$

(3.2) $$\underline{D}_{\mathbf{N}}(\psi \mid \mu : x) = \sup_{\mathbf{S} \in \mathfrak{A}} \inf \{\psi(I, x)/\mu(I, x) : (I, x) \in \mathbf{S} \cap \mathbf{N}\},$$

where these always exist as extended real numbers provided we intepret $c/0 = +\infty$ $(c > 0)$, $c/0 = -\infty$ $(c < 0)$ and $0/0 = 0$. If in the above $\bar{D}_{\mathbf{N}}(\psi \mid \mu : x) = \underline{D}_{\mathbf{N}}(\psi \mid \mu : x)$ we say the *derivative* of ψ with respect to μ (and $\mathbf{N}$) exists at x and we write the common value as $D_{\mathbf{N}}(\psi \mid \mu : x)$.

The first result asserts that the indefinite integral has a derivative equal to the integrand everywhere excepting a null set with respect to the $(\mu, \mathbf{N})$-inner variation. The proof is essentially due to Henstock.

THEOREM 3.3. *Suppose that $(T, \mathfrak{A}, \mathbf{I})$ is decomposable. If f is a real-valued function on a set $X \subseteq T$ and $V(\psi - f\mu, \mathfrak{A}[X]) = 0$ then $D_{\mathbf{N}}(\psi \mid \mu : x) = f(x)$ everywhere in X excepting a set of $(\mu, \mathbf{N})$-inner variation zero. In particular if $F = \int f\mu$ exists in $(T, \mathfrak{A}, \mathbf{I})$ we have $D_{\mathbf{N}}(F \mid \mu : x) = f(x)$ almost everywhere with respect to the $(\mu, \mathbf{N})$-inner variation.*

Proof. Let X_0 denote the set of points in X at which $D_{\mathbf{N}}(\psi \mid \mu : x)$ either does not exist or does not equal $f(x)$. For each $x \in X_0$ we choose $\varepsilon(x)$ so that $0 < \varepsilon(x) < 1$ and so that

$$|\psi(I, x) - f(x)\,\mu(I, x)| \geqq \varepsilon(x)|\mu(I, x)|, \tag{3.3.1}$$

for at least one $(I, x) \in \mathbf{S} \cap \mathbf{N}$ for every $\mathbf{S} \in \mathfrak{A}[X_0]$ otherwise the derivative would exist and equal $f(x)$.

Define the sets $X_k = \{x \in X_0 : 2^{-k} \leqq \varepsilon(x) < 2^{1-k}\}$ so that $\bigcup_{k=1}^{\infty} X_k = X_0$ and the $\{X_k\}$ are disjoint. If $\varepsilon > 0$ is given we choose $\mathbf{S}_k \in \mathfrak{A}$ so that $V(\psi - f\mu, \mathbf{S}_k[X]) < \varepsilon/4^k$ and then choose $\mathbf{S} \in \mathfrak{A}$ so that $\mathbf{S}[X_k] \subseteq \mathbf{S}_k[X_k]$.

Let $\mathbf{F}$ be the collection of $(I, x) \in \mathbf{S} \cap \mathbf{N}[X_0]$ for which (3.3.1) holds. By our hypotheses $\mathbf{F}$ is an $\mathbf{N}$-cover of X_0 so $IV_{\mathbf{N}}(\mu, X_0) \leqq V(\mu, \mathbf{F})$. If now $\mathbf{D} \subseteq \mathbf{F}$ is a division we have

$$\begin{aligned}(\mathbf{D})\sum |\mu(I, x)| &\leqq \sum_{k=1}^{\infty} 2^k (\mathbf{D}[X_k]) \sum \varepsilon(x)|\,\mu(I, x)| \\ &\leqq \sum_{k=1}^{\infty} 2^k (\mathbf{D}[X_k]) \sum |\psi(I, x) - f(x)\,\mu(I, x)| \\ &\leqq \sum_{k=1}^{\infty} 2^k\, V(\psi - f\mu, \mathbf{S}_k[X_k]) \leqq \varepsilon,\end{aligned}$$

and hence $IV_{\mathbf{N}}(\mu, X_0) \leqq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $IV_{\mathbf{N}}(\mu, X_0) = 0$ as required.

The two theorems which now follow assert various properties of the derivates. We first prove a lemma related to a classical one in Saks [**4**; p. 114].

LEMMA 3.4. *If $|\bar{D}_{\mathbf{N}}(\psi \mid \mu : x)| \geqq \alpha > 0$ at every point of a set X then*

$$\alpha IV_{\mathbf{N}}(\mu, X) \leqq \psi^*(X).$$

Proof. Let β be an arbitrary number with $0 < \beta < \alpha$ and let $\mathbf{S} \in \mathfrak{A}$: if $\mathbf{F}$ denotes the collection of all $(I, x) \subseteq \mathbf{S} \cap \mathbf{N}[X]$ for which $|\psi(I, x)/\mu(I, x)| \geqq \beta$ then our assumptions imply that $\mathbf{F}$ is an $\mathbf{N}$-cover of X. Hence

$$\beta IV_{\mathbf{N}}(\mu, X) \leqq \beta V(\mu, \mathbf{F}) \leqq V(\psi, \mathbf{F}) \leqq V(\psi, \mathbf{S}[X]),$$

so that $\beta IV_{\mathbf{N}}(\mu, X) \leqq \psi^*(X)$ and, on letting β increase to α, the assertion of the lemma follows.

THEOREM 3.5. *If ψ has σ-finite variation then the set of points at which either of the derivates $\bar{D}_{\mathbf{N}}(\psi \mid \mu : x)$ or $\underline{D}_{\mathbf{N}}(\psi \mid \mu : x)$ is infinite has $(\mu, \mathbf{N})$-inner variation zero.*

Proof. We need consider only the set $X_\infty = \{x : |\bar{D}_{\mathbf{N}}(\psi|\mu : x)| = +\infty\}$ as $-\underline{D}_{\mathbf{N}}(\psi \mid \mu : x) = \bar{D}_{\mathbf{N}}(-\psi \mid \mu : x)$ would give the corresponding result for the lower derivate.

Let $X_m = \{x : |\bar{D}_{\mathbf{N}}(\psi \mid \mu : x)| \geqq m\}$ then by the lemma

$$mIV_{\mathbf{N}}(\mu, X_\infty) \leqq mIV_{\mathbf{N}}(\mu, X_m) \leqq \psi^*(X_m).$$

If ψ has finite variation then on letting $m \to \infty$, we obtain $IV_{\mathbf{N}}(\mu, X_\infty) = 0$; if ψ has finite variation on a sequence of sets covering T, the intersection of X_∞ with each such set has $(\mu, \mathbf{N})$-inner variation zero and (2.4) then gives the final result.

THEOREM 3.6. *If ψ has zero variation on a set X, $D_{\mathbf{N}}(\psi \mid \mu : x) = 0$ for all x in X excepting a set of $(\mu, \mathbf{N})$-inner variation zero.*

Proof. Let $X_m = \{x \in X\colon |\bar{D}_{\mathbf{N}}(\psi \mid \mu : x)| \geqq 1/m\}$ and let

$$X_0 = \{x \in X\colon \bar{D}_{\mathbf{N}}(\psi \mid \mu : x) \neq 0\}.$$

Applying Lemma 3.4 we get $1/mIV_{\mathbf{N}}(\mu, X_m) \leqq \psi^*(X_m) \leqq \psi^*(X) = 0$, so that, using (2.4), $IV_{\mathbf{N}}(\mu, X_0) \leqq \sum_{m=1}^{\infty} IV_{\mathbf{N}}(\mu, X_m) = 0$. Similar results apply to the set at which $\underline{D}_{\mathbf{N}}(\psi \mid \mu : x) \neq 0$ which proves the theorem.

THEOREM 3.7. *If μ has σ-finite variation on a set X and if $f(x) = \bar{D}_{\mathbf{N}}(\psi \mid \mu : x)$ is finite for every x in X then $IV_{\mathbf{N}}(\psi - f\mu, X) = 0$.*

Proof. Let $\varepsilon > 0$ and $\mathbf{S} \in \mathfrak{A}$: the set $\mathbf{F}$ of all $(I, x) \in \mathbf{S} \cap \mathbf{N}[X]$ for which $f(x) - \varepsilon \leqq \psi(I, x)/\mu(I, x) \leqq f(x) + \varepsilon$ is an $\mathbf{N}$-cover of X. Hence

$$IV_{\mathbf{N}}(\psi - f\mu, X) \leqq V(\psi - f\mu, \mathbf{F})$$

and, if $\mathbf{D} \subseteq \mathbf{F}$ is a division,

$$(\mathbf{D})\sum |\psi(I, x) - f(x)\,\mu(I, x)| \leqq (\mathbf{D})\sum \varepsilon\,|\mu(I, x)|$$
$$\leqq \varepsilon V(\mu, \mathbf{S}[X]),$$

so that $IV_{\mathbf{N}}(\psi - f\mu, X) \leqq \varepsilon\mu^*(X)$.

If $\mu^*(X) < +\infty$, the result follows immediately; if μ has σ-finite variation on X, the result follows in the usual manner.

COROLLARY 3.8. *If μ has σ-finite variation on a set X and if $f(x) = \bar{D}_{\mathbf{N}}(\psi \mid \mu : x)$ is finite for every x in X, $IV_{\mathbf{N}}(f\mu, X) \leqq \psi^*(X)$.*

Proof. By (2.3)

$$IV_{\mathbf{N}}(f\mu, X) \leqq IV_{\mathbf{N}}(\psi - f\mu, X) + \psi^*(X),$$

and the result then follows from the theorem.

4. *Additional remarks*

The investigation of the present theory under stronger hypotheses will be carried out in a later paper; it does not seem possible to obtain many further results in this general setting. In particular we mention the following requirements.

The notion of a division system is quite weak: a division system $(T, \mathfrak{A}, \mathbf{I})$ in which every $\mathrm{S} \in \mathfrak{A}$ contains a division of every elementary set is said to be a *division space*. In such spaces there is a fully developed theory of integration [**2, 3**] and certain of the classical differentiation results for the Lebesgue integral can be extended. Moreover it is usually convenient to have a topology present in T: we say a division system $(T, \mathfrak{A}, \mathbf{I})$ is compatible with the topology if for every open set G there is an $\mathrm{S} \in \mathfrak{A}$ so that $\mathrm{S}[G] \subseteq \mathrm{S}(G)$. In the presence of this extra structure more specific results are obtainable.

References

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Simon Fraser University,
B.C., Canada.