COVERING SYSTEMS AND DERIVATES IN HENSTOCK DIVISION SPACES II

B. S. THOMSON

In most theories of covering systems and derivates some attempt is made to obtain results analogous to the classical Vitali theorem; within the setting of the theory formulated in the first paper **[2]** it appears that Vitali properties are best obtained by a modification of the methods of A. P. Morse's derivation theory. In this paper we obtain a Vitali property on the Henstock division systems by adopting an appropriate halo assumption (for the standard theory see **[1**; Chapter IV]).

The notation and terminology throughout will be identical with that in **[2]**.

If $(T, \mathfrak{A}, \mathbf{I})$ is a division system, N a covering system on $(T, \mathfrak{A}, \mathbf{I})$ and μ a real-valued function on $\mathbf{I}$ we have defined three set functions, usually outer measures:

$$\mu^*(X) = \inf_{\mathbf{S} \in \mathfrak{A}} V(\mu, \mathbf{S}[X])$$

$$_{\mathbf{N}}\mu^*(X) = \inf_{\mathbf{S} \in \mathfrak{A}} V(\mu, \mathbf{S} \cap \mathbf{N}[X])$$

$$_{\mathbf{N}}\mu_*(X) = IV_{\mathbf{N}}(\mu, X).$$

We shall be interested in obtaining Vitali conditions on $_{\mathbf{N}}\mu^*$ for, as was seen in **[2]**, this can serve to establish the relationship between $_{\mathbf{N}}\mu^*$ and $_{\mathbf{N}}\mu_*$. For this purpose it is convenient to introduce the division system $(T, \mathfrak{A}_{\mathbf{N}}, \mathbf{N})$ where

$$\mathfrak{A}_{\mathbf{N}} = \{\mathbf{S} \cap \mathbf{N} : \mathbf{S} \in \mathfrak{A}\}.$$

We need also the following definitions.

Definition 1. For any bounded non-negative function Δ on N and any $\alpha > 1$ we define

$$H_\Delta^{\alpha}(I, x) = \bigcup\{J : (J, y) \in \mathbf{N}, I \cap J \neq \emptyset \quad \text{and} \quad \Delta(J, y) \leqslant \alpha\Delta(I, x)\}.$$

Definition 2. A covering system N is said to satisfy a *halo condition* with respect to μ on a division system $(T, \mathfrak{A}_{\mathbf{N}}, \mathbf{N})$ if there are real numbers $\alpha > 1$ and $\lambda > 0$, and a bounded non-negative function Δ on N such that for all $(I, x) \in \mathbf{N}$ and $\mathbf{S} \in \mathfrak{A}_{\mathbf{N}}$

$$V\Big(\mu, \mathbf{S}(H_\Delta^{\alpha}(I, x))\Big) \leqslant \lambda V(\mu, \mathbf{S}(I)).$$

THEOREM 1. *Let* $(T, \mathfrak{A}, \mathbf{I})$ *be a division system,* N *a covering system on* $(T, \mathfrak{A}, \mathbf{I})$, μ *a real-valued function on* N *and* $X \subset T$. *Suppose that*

(i) *for every* $(I, x) \in \mathbf{N}$ *there is an* $\mathbf{S} \in \mathfrak{A}_{\mathbf{N}}$ *with* $\mathbf{S}[\diagdown I] \subset \mathbf{S}(\diagdown I)$,

(ii) N *satisfies a halo condition in* $(T, \mathfrak{A}_{\mathbf{N}}, \mathbf{N})$ *with respect to* μ, *and*

(iii) *there is a set* $G \subset T$ *and* $\mathbf{S}_0 \in \mathfrak{A}_{\mathbf{N}}$ *so that* $\mathbf{S}_0[X] \subset \mathbf{S}_0(G)$ *and* $V(\mu, \mathbf{S}_0(G)) < +\infty$.

Received 21 December 1973.

[J. LONDON MATH. SOC. (2), 10 (1975), 125–128]

Then there exists a sequence $(I_k, x_k) \subset \mathbf{N}$ with the $\{I_k\}$ disjoint such that

$$_{\mathbf{N}}\mu^*\left(X \setminus \bigcup_{k=1}^{\infty} I_k\right) = 0.$$

Proof. The construction of the sequence (I_k, x_k) is based on a standard technique for halo conditions (*cf.* [**1**]). Let $\mathbf{S}_0$ and G be chosen as in (iii) and suppose that Δ, α and λ are from the halo condition (Definition 2). We begin by setting

$$\mathbf{J}_1 = \{(J, y) \in \mathbf{S}_0[X] : \mu(J, y) \neq 0\},\ \Delta_1 = \sup\{\Delta(J, y) : (J, y) \in \mathbf{J}_1\}$$

and choosing $(J_1, y_1) \in \mathbf{J}_1$ with $\Delta(J_1, y_1) \geqslant \Delta_1/\alpha$. Since Δ is bounded and $\alpha > 1$ this is possible unless $\mathbf{J}_1 = \varnothing$ in which case $_{\mathbf{N}}\mu^*(X) = 0$ and the theorem is trivially proved.

For any ordinal γ we continue inductively be defining

$$\mathbf{J}_\gamma = \{(J, y) : (J, y) \in \mathbf{S}_0[X],\quad \mu(J, y) \neq 0 \quad\text{and}\quad J \cap J_{\gamma'} = \varnothing \quad \text{for all} \quad \gamma' < \gamma\},$$

$$\Delta_\gamma = \sup\{\Delta(J, y) : (J, y) \in \mathbf{J}_\gamma\}$$

and choose $(J_\gamma, y_\gamma) \in \mathbf{J}_\gamma$ so that $\Delta(J_\gamma, y_\gamma) \geqslant \Delta_\gamma/\alpha$; this is possible up to the point at which $\mathbf{J}_\gamma = \varnothing$ in which case the process terminates.

Since the sets in the sequence $\{J_\gamma\}$ are disjoint we have for any non-empty finite set of indices γ that

$$0 < \sum |\mu(J_\gamma, y_\gamma)| \leqslant V(\mu, \mathbf{S}_0(G)) < +\infty$$

so that $\{(J_\gamma, x_\gamma)\}$ is at most countable.

Let $\{(I_k, x_k)\}$ be a rearrangement of the possibly transfinite sequence $\{(J_\gamma, y_\gamma)\}$ into a conventional sequence: we shall prove the theorem by establishing that

$$_{\mathbf{N}}\mu^*\left(X \setminus \bigcup_1^{\infty} I_k\right) = 0.$$

For each natural number n let $\mathbf{S}_n \in \mathfrak{A}_{\mathbf{N}}$ be chosen so that

$$\mathbf{S}_n \subset \mathbf{S}_0 \quad\text{and}\quad \mathbf{S}_n\left[\setminus \bigcup_1^n I_k\right] \subset \mathbf{S}_n\left(\setminus \bigcup_1^n I_k\right);$$

this is possible by repeated applications of (i).

Suppose that (J, y) belongs to

$$\mathbf{S}_n\left[\setminus \bigcup_1^{\infty} I_k\right]:$$

then either $\mu(J, y) = 0$ or else J intersects some member of $\{J_\gamma\}$. Let γ^* be the first ordinal for which $J_{\gamma^*} \cap J \neq \varnothing$ and let i^* be the relabelling for γ^*. Then $(J, y) \in \mathbf{J}_{\gamma^*}$ and so $\Delta(J, y) \leqslant \Delta_{\gamma^*} \leqslant \alpha\Delta(J_{\gamma^*}, y_{\gamma^*})$ from which it follows that $(J, y) \in \mathbf{S}_n(H_\Delta{}^\alpha(J_{\gamma^*}, y_{\gamma^*}))$.

But from the construction of $\mathbf{S}_n$, J cannot intersect any of $I_1, \ldots$ to I_n, so that $i^* > n$.

Considering then an arbitrary division

$$\mathbf{D} \subseteq \mathbf{S}_n\left[X \setminus \bigcup_1^\infty I_k\right],$$

we have from the above arguments that

$$(\mathbf{D})\sum|\mu(J, y)| \leqslant \sum_{k>n} V\left(\mu, \mathbf{S}_n(H_\Delta^{\alpha}(I_k, x_k))\right) \leqslant \sum_{k>n} V\left(\mu, \mathbf{S}_0(H_\Delta^{\alpha}(I_k, x_k))\right)$$

and, using the halo condition, this is less than $\lambda\sum_{k>n} V(\mu, \mathbf{S}_0(I_k))$. Thus

$${}_{\mathbf{N}}\mu^*\left(X \setminus \bigcup_1^\infty I_k\right) \leqslant V\left(\mu, \mathbf{S}_n\left[X \setminus \bigcup_1^\infty I_k\right]\right) \leqslant \lambda\sum_{k>n} V(\mu, \mathbf{S}_0(I_k))$$

for all n; but

$$\sum_1^\infty V(\mu, \mathbf{S}_0(I_k)) \leqslant V(\mu, \mathbf{S}_0(G)) < +\infty$$

and so letting $n \to \infty$ in the previous inequality yields

$${}_{\mathbf{N}}\mu^*\left(X \setminus \bigcup_1^\infty I_k\right) = 0$$

as required.

Condition (i) of the theorem restricts the structure of the system $(T, \mathfrak{A}, \mathbf{I})$ without restricting μ: it is possible to give a condition which relaxes this somewhat while putting more control over μ. For example, it can be easily shown that (i)′ given here can be substituted in the statement of the theorem.

(i)′ *μ is regular* [3; Definition 3] *in* $(T, \mathfrak{A}_{\mathbf{N}}, \mathbf{N})$ *and for every division* $\mathbf{D} \subset \mathbf{N}$ *with* $E = \sigma(\mathbf{D})$,

$$V(\mu, \mathfrak{A}_{\mathbf{N}}(E)[\setminus E]) = 0.$$

A somewhat different version of the theorem can be stated in those situations where T has a topology. As in [2] we say that $(T, \mathfrak{A}, \mathbf{I})$ is *compatible* with the topology if for every open set G in T there is an $\mathbf{S} \in \mathfrak{A}$ such that $\mathbf{S}[G] \subset \mathbf{S}(G)$. In this setting Theorem 1 assumes a similar but more familiar form.

THEOREM 2. *Let T be a topological space and suppose that $(T, \mathfrak{A}, \mathbf{I})$ is a division system compatible with the topology on T. If μ is a real-valued function on $\mathbf{I}$, $X \subset T$, $\mathbf{N}$ is a covering system on $(T, \mathfrak{A}, \mathbf{I})$ and*

(i) *for every $(I, x) \in \mathbf{N}$, I is closed and $x \in I$;*

(ii) $\mathbf{N}$ *satisfies a halo condition on $(T, \mathfrak{A}_{\mathbf{N}}, \mathbf{N})$ with respect to μ, and*

(iii) *there is an open set $G \supset X$ with ${}_{\mathbf{N}}\mu^*(G) < +\infty$,*

then there exists a sequence $\{(I_k, x_k)\} \subset \mathbf{N}$ with the $\{I_k\}$ disjoint such that

$${}_{\mathbf{N}}\mu^*\left(X \setminus \bigcup_1^\infty I_k\right) = 0.$$

Proof. It is enough to observe that such a system evidently satisfies the hypotheses of Theorem 1.

We conclude by remarking on the problem of identifying the measures ${}_N\mu^*$ and ${}_N\mu_*$: in general these will be distinct. The system of pairs (I, x) where I is a closed interval in $\mathbf{R}^2$ and $x \in I$ can be used to obtain an example in which ${}_N\mu^*$ is Lebesgue measure in $\mathbf{R}^2$ while ${}_N\mu_*$ vanishes everywhere. However, Theorems 1 and 2 can be used in combination with [**2**; Theorem 2.5] to obtain certain conditions under which ${}_N\mu^* \equiv {}_N\mu_*$.

References

1. C. Hayes and C. Pauc, " Derivation and martingales ", *Ergebnisse der Math.* (Springer).
2. B. S. Thomson, " Covering systems and derivates in Henstock division spaces ", *J. London Math. Soc.*, 4 (1971), 1C3–108.
3. ———, " A theory of integration ", *Duke Math. J.*, 39 (1972), 503–510.

Department of Mathematics,
Simon Fraser University,
Burnaby 2,
B.C., Canada.