CONSTRUCTION OF MEASURES IN METRIC SPACES

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An arbitrary non-negative set function τ defined on a family $\mathbf{C}$ of subsets of a metric space Ω (with $\varnothing \in \mathbf{C}$ and $\tau\varnothing = 0$) can be used to generate outer measures on Ω:

for $E \subset \Omega$ and $\delta > 0$ define

$$^1\mu_\tau E = \inf \left\{ \sum_{i=1}^{\infty} \tau C_i : C_i \in \mathbf{C}, \quad \bigcup_{i=1}^{\infty} C_i \supset E \right\},$$

$$^2\mu_{\tau\delta} E = \inf \left\{ \sum_{i=1}^{\infty} \tau C_i : C_i \in \mathbf{C}, \quad \bigcup_{i=1}^{\infty} C_i \supset E \quad \text{and} \quad \operatorname{diam} C_i \leqslant \delta \right\},$$

$$^2\mu_\tau E = \lim_{\delta \to 0} {}^2\mu_{\tau\delta} E.$$

These constructions of $^1\mu_\tau$ and $^2\mu_\tau$ from τ are originally due to Carathéodory and Hausdorff and are now, following Munroe [**4**], commonly referred to as Methods I and II respectively. Both constructions generate outer or Carathéodory measures but the latter yields a measure which has many properties compatible with the topology on Ω.

In this paper we present two further constructions of measures from such arbitrary set functions, constructions which share the desirable topological features of Method II. These constructions arise from ideas of " packing ", in contrast to Methods I and II which are essentially approximations by " coverings "; what we term Method III is due to R. Henstock [**2**], while Method IV arises from Henstock's [**1**] notion of inner variation (which is not, however, a measure).

1. *Notation and results*

The terminology is modelled after Rogers [**3**] whose first chapter serves as an excellent introduction to the theory of outer or Carathéodory measures.

1.1. A set function τ defined on a class $\mathbf{C}$ of subsets of a set Ω is called a *premeasure* if $\varnothing \in \mathbf{C}$, $0 \leqslant \tau C \leqslant +\infty$ for $C \in \mathbf{C}$, and $\tau\varnothing = 0$.

1.2. A premeasure μ defined on all subsets of Ω with the property $\mu A \leqslant \sum_{i=1}^{\infty} \mu B_i$ if $A \subset \bigcup_{i=1}^{\infty} B_i$ is called a *measure.*

1.3. A measure μ defined on a metric space Ω with metric ρ is said to be a *metric measure* if $\mu(A \cup B) = \mu A + \mu B$ whenever $\inf\{\rho(x, y) : x \in A, y \in B\} > 0$.

1.4. Given a premeasure τ with domain $\mathbf{C}$ and a set $\mathbf{I} \subset \mathbf{C} \times \Omega$, $\bar{\tau}(\mathbf{I}, E)$ denotes the

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supremum of the non-negative numbers

$$\left\{\sum_{i=1}^{n} \tau C_i : (C_i, x_i) \in \mathbf{I},\ x_i \in E \text{ and } \{C_i\}_{i=1}^{n} \text{ pairwise disjoint}\right\}$$

(for convenience, as we are considering only non-negative numbers, sup $\varnothing = 0$}.

1.5. Given a function $\eta : \Omega \to 2^{\Omega}$ we denote $\mathbf{C} \times \Omega_\eta$ as the set of all $(I, x) \in C \times \Omega$ with $x \in I \subset \eta(x)$. (Here 2^{Ω} is the class of all subsets of Ω.)

1.6. A set function ${}^3\mu_\tau$ is said to have been constructed from the premeasure τ on a metric space Ω by *Method III* if for $E \subset \Omega$

$${}^3\mu_\tau E = \inf_{\eta} \bar{\tau}(\mathbf{C} \times \Omega_\eta, E)$$

where the infimum is with regard to all functions η on Ω which map each $x \in \Omega$ to a neighbourhood of x.

1.7. A subset N of $\mathbf{C} \times \Omega$ is said to *cover* $E \subset \Omega$ *finely* if for each $x \in E$ and each neighbourhood $\eta(x)$ of x there is an $(I, x) \in \mathrm{N}$ with $x \in I \subset \eta(x)$.

1.8. A set function ${}^4\mu_\tau$ is said to have been constructed from the premeasure τ on a metric space Ω by *Method IV* if for $E \subset \Omega$

$${}^4\mu_\tau E = \inf_{\mathrm{N}} \bar{\tau}(\mathrm{N}, E)$$

where the infimum is with regard to all $\mathrm{N} \subset \mathbf{C} \times \Omega$ that cover E finely (regarding as usual $\inf \varnothing = +\infty$).

It is well known that the Method II construction invariably yields a metric measure; we prove this same fact for Methods III and IV.

THEOREM 1. *Let τ be a premeasure on a metric space Ω; then the set functions constructed from τ by Methods III and IV are metric measures on Ω.*

Proof. Let ${}^3\mu_\tau$ be the set function constructed from τ by Method III. Clearly ${}^3\mu_\tau$ is at least a premeasure; suppose $A \subset \bigcup_{i=1}^{\infty} B_i$ and $\varepsilon > 0$. We may choose a function $\eta_i : x \to$ neighbourhood of x so that for each i

$$\bar{\tau}(\mathbf{C} \times \Omega_{\eta_i}, B_i) \leqslant {}^3\mu_\tau B_i + \varepsilon/2^i.$$

Then a further function η can be chosen so that if $x \in B_i \setminus B_{i-1}$ (using $B_0 = \varnothing$) then $\eta(x_i) \subset \eta_i(x_i)$. Now one can verify that

$$\begin{aligned} {}^3\mu_\tau A \leqslant \bar{\tau}(\mathbf{C} \times \Omega_\eta, A) &\leqslant \sum_{i=1}^{\infty} \bar{\tau}(\mathbf{C} \times \Omega_{\eta_i}, B_i \setminus B_{i-1}) \\ &\leqslant \sum \bar{\tau}(\mathbf{C} \times \Omega_{\eta_i}, B_i) \\ &\leqslant \sum {}^3\mu_\tau B_i + \varepsilon. \end{aligned}$$

Thus whenever $A \subset \bigcup_{i=1}^{\infty} B_i$ it follows that ${}^3\mu_\tau A \leqslant \sum_i {}^3\mu_\tau B_i$ and ${}^3\mu_\tau$ is a measure.

Similarly, to show that ${}^4\mu_\tau$ constructed from τ by Method IV is a measure observe that ${}^4\mu_\tau$ is a premeasure and suppose $A \subset \bigcup_{i=1}^{\infty} B_i$. Let $\varepsilon > 0$ and choose $\mathrm{N}_i \subset \mathbf{C} \times \Omega$ a fine cover of B_i so that $\bar{\tau}(\mathbf{N}_i, B_i) \leqslant {}^4\mu_\tau B_i + \varepsilon/2^i$ (if no such $\mathbf{N}_i$ exists then there is nothing to prove). Let $\mathbf{N} = \bigcup_{i=1}^{\infty} \mathbf{N}_i$ and clearly $\mathbf{N}$ is a fine cover of A, so that

$$
\begin{aligned}
{}^4\mu_\tau A \leqslant \bar{\tau}(\mathbf{N}, A) &\leqslant \sum_{i=1}^{\infty} \bar{\tau}(\mathbf{N}_i, B_i) \\
&\leqslant \sum_{i=1}^{\infty} {}^4\mu_\tau B_i + \varepsilon
\end{aligned}
$$

and as before ${}^4\mu_\tau$ is a measure on Ω.

To show that these are metric measures, suppose $A, B \subset \Omega$ with $\inf\{\rho(x, y) : x \in A, y \in B\} > 0$; then there exist open sets $G_1 \supset A$, $G_2 \supset B$ with $G_1 \cap G_2 = \varnothing$. Let η be any function mapping $x \in \Omega$ to a neighbourhood of x with $\eta(x) \subset G_i$ if $x \in G_i$ $(i = 1, 2)$; then clearly

$$\bar{\tau}(\mathbf{C} \times \Omega_\eta, A \cup B) = \bar{\tau}(\mathbf{C} \times \Omega_\eta, A) + \bar{\tau}(\mathbf{C} \times \Omega_\eta, B),$$

and from this it can be seen that

$${}^3\mu_\tau(A \cup B) = {}^3\mu_\tau A + {}^3\mu_\tau B$$

as required. Similar arguments apply to ${}^4\mu_\tau$.

The measures ${}^3\mu_\tau$ and ${}^4\mu_\tau$ need not be comparable, simply because there may be no fine covers of a given set. If, however, $\mathbf{C} \times \Omega$ is itself a fine cover of Ω, then it is easy to show that ${}^3\mu_\tau \geqslant {}^4\mu_\tau$; §2 below mentions examples illustrating that equality can occur but fails in general. Note, moreover, that the theorem can be extended to more general settings than metric spaces (e.g. that of Bledsoe and Morse [5]) since the construction is based on the topology in Ω rather than the metric.

A premeasure is quite a general set function and these Methods I–IV smooth its values to a measure which need not have a great deal to do with the original premeasure. A question which naturally arises then is what happens when τ is itself a measure or the restriction of a measure. Rogers [3] illustrates the case for Methods I and II and the next theorem gives a result which shows what can happen with Method III.

THEOREM 2. *Let ν^* be a $\mathscr{G}_\delta$-regular metric measure on a separable metric space Ω and let ν denote the restriction of ν^* to the closed subsets of Ω. Then ν is a premeasure and ${}^3\mu_\nu$ and ν^* coincide on sets of finite ν^*-measure.*

Proof. We show firstly that in general ${}^3\mu_\nu E \leqslant \nu^* E$ for $E \subset \Omega$. If $\nu^* E = +\infty$ there is nothing to prove, while if $\nu^* E < +\infty$ there is, for any $\varepsilon > 0$, an open set $G \supset E$ with $\nu^* G < \nu^* E + \varepsilon$. For any function $\eta : \Omega \to 2^\Omega$ with $\eta(x)$ a neighbourhood of x and $\eta(x) \subset G$ for all $x \in G$, it is evident that

$${}^3\mu_\nu E \leqslant \bar{\nu}(\mathbf{C} \times \Omega_\eta, E) \leqslant \nu^* G \leqslant \nu^* E + \varepsilon$$

and, as $\varepsilon > 0$ is arbitrary, the inequality follows.

For the reverse inequality $\nu^* E \leqslant {}^3\mu_\nu E$, which we need only prove for $0 < \nu^* E < +\infty$, let G and η be as above. Consider the class $\mathfrak{P}$ of all subsets $\mathbf{P} \subset \mathbf{C} \times \Omega_\eta$, where $\mathbf{C}$ denotes the closed subsets of Ω, with the properties (i) $\nu I > 0$ if $(I, x) \in \mathbf{P}$, (ii) $I \cap J = \varnothing$ if $(I, x), (J, y) \in \mathbf{P}$ and $(I, x) \neq (J, y)$. $\mathfrak{P}$ is partially ordered by set inclusion, non-empty (this can be obtained by repeating the arguments we use below to establish maximality) and satisfies the hypotheses of the Hausdorff maximality principle. Let $\mathbf{P}_0$ be a maximal element of $\mathfrak{P}$.

Clearly $\mathbf{P}_0$ is at most countable since $\sum\{\nu I : (I, x) \in \mathbf{P}_0\} < \nu^* G < +\infty$; so suppose $\mathbf{P}_0 = \{(I_i, x_i)\}_{i=1}^{\infty}$. We claim that $\nu^*\left(E \setminus \bigcup_1^{\infty} I_i\right) = 0$. If not, then $F = E \setminus \bigcup_1^{\infty} I_i$ has positive ν^*-measure and then one at least of the sets $\{F \cap \eta(x) : x \in F\}$ has positive measure; this would be obvious if $\{F \cap \eta(x) : x \in F\}$ were countable, but as, however, we are in a separable metric space this family may be reduced to a countable one covering the same set (e.g. Kelley [**6**; p. 49]). Thus there is an $x_0 \in F$ with $\nu^*(F \cap \eta(x_0)) > 0$. Since ν^* is $\mathscr{G}_\delta$ regular [**3**; Theorem 22], there is a closed set $C \subset F \cap \eta(x_0)$ with positive ν^*-measure. Write $I_0 = C \cup \{x_0\}$; then $\mathbf{P}' = \{(I_i, x_i)\}_{i=0}^{\infty}$ belongs to $\mathfrak{P}$ and contradicts the maximality of $\mathbf{P}_0$. Hence we must have that $\nu^*\left(E \setminus \bigcup_1^{\infty} I_i\right) = 0$.

Computing from this, then

$$\nu^* E \leqslant \nu^*\left(E \setminus \bigcup_1^{\infty} I_i\right) + \sum_{i=1}^{\infty} \nu I_i$$

$$\leqslant \bar{\nu}(\mathbf{C} \times \Omega_\eta, E)$$

and finally $\nu^* E \leqslant {}^3\mu_\nu E$ as required.

2. *Examples*

In this section we briefly collect some examples to illustrate the preceeding.

A trivial example shows that some such restrictions as in Theorem 2 are needed. Let Ω be an uncountable metric space with the discrete metric and write $\nu E = 0$ or $\alpha (0 < \alpha \leqslant +\infty)$ according as E is countable or uncountable. Then ν is a measure on Ω, but ${}^3\mu_\nu$ vanishes everywhere.

For a very classical example one can take in n-dimensional Euclidean space R^n the class $\mathbf{C}$ of all open rectangles $R(a, b)$ where $R(a, b) = \{x = (x_1, x_2, \ldots, x_n) : a_i < x_i < b_i\}$ where $a = (a_1, a_2, \ldots, a_n)$, $b = (b_1, b_2, \ldots, b_n)$ and $a_i \leqslant b_i$ $(i = 1, 2, \ldots, n)$. If λ is defined on $\mathbf{C}$ by $\lambda R(a, b) = \prod_{i=1}^{n} (b_i - a_i)$, then λ is a premeasure. Rogers [**3**] proves the well-known result that Methods I and II both yield Lebesgue measure; an application of the Vitali Theorem will show that Method III does as well. For $n \geqslant 2$, however, it can be shown that the Method IV construction vanishes everywhere (e.g. Munroe [**4**; §39] establishes that given $\varepsilon > 0$ and an interval $[c, d]$ in R^2 there is a fine cover $\mathrm{N} \subset \mathbf{C} \times R^2$ of $[c, d]$ with $\bar{\lambda}(\mathrm{N}, [c, d]) < \varepsilon$). For $n = 1$ of course the Method IV construction yields classical Lebesgue linear measure.

References

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