ON THE DERIVED NUMBERS OF VBG_* FUNCTIONS

B. S. THOMSON

The behaviour of the derived numbers of a function may give little information about the nature of the function itself. Indeed a typical continuous function, in the sense of Bruckner [1; p. 213], has at every point every extended real number as a derived number. However, with a regularity condition imposed, such as that the function be VBG_*, the connection is more direct. It is for this reason that a number of theorems which draw inferences on the nature of a function from information about the derived numbers require hypotheses that force the function to be VBG_* on some set.

In this article we study some methods that are useful in obtaining such inferences and which seem to reveal the reasons why such theorems should exist. While most of the results are known the methods we introduce have not received any great attention.

The technique consists of introducing a pair of measures Δf^* and Δf_* associated with any real valued function f; these measures describe the total variation of the function and carry information about the derivative and about the derived numbers of f. The deep part of the theory is to provide the relations that hold between the measures; the rest is more or less easy and gives a unified way of looking at a great many classical results and classical concepts. For example the properties VBG_* and ACG_* of Saks [8] can be expressed directly by the measure Δf^*. Other studies of these measures in a more abstract setting have been undertaken by McGill [6] and recently by Henstock [5].

2. *Basic definitions*

We restrict attention throughout to a closed interval $[a, b]$: m will denote Lebesgue measure on that interval, $C[a, b]$ the usual space of continuous functions on $[a, b]$, and the terms VB_*, VBG_*, AC_*, ACG_* will be used precisely as in Saks [8]. If f is a real valued function on $[a, b]$ then $f'(x)$, $D^+f(x)$, $D_+f(x)$, $D^-f(x)$, and $D_-f(x)$ denote the derivative (if it exists) and the four Dini derivatives (which do exist) of the function f at the point x with the usual conventions if x happens to be an endpoint. We need as well a notation for the derived numbers of the function f: write $\mathrm{der}^+f(x)$ as the set of limit points of the expression $(f(x+h)-f(x))/h$ as $h \to 0+$ and $\mathrm{der}^-f(x)$ as the set of derived numbers from the other side. The set $\mathrm{der}\, f(x)$ is the union of the two sets $\mathrm{der}^+ f(x)$ and $\mathrm{der}^- f(x)$. If f is continuous then in fact $\mathrm{der}^+ f(x)$ is precisely the interval $[D_+f(x), D^+f(x)]$ and $\mathrm{der}^- f(x)$ is the interval $[D_-f(x), D^-f(x)]$.

The study of the derived numbers or the Dini derivatives leads immediately to the following concepts.

Definition. 2.1. A subset $\mathscr{C}$ of $\mathscr{I} \times [a, b]$ (where here and elsewhere $\mathscr{I}$ denotes the collection of all closed subintervals of $[a, b]$) is said to be a *full cover* of a set

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$X \subseteq [a, b]$ if for every $x \in X$ there exists a $\delta > 0$ such that (I, x) belongs to $\mathscr{C}$ whenever $I \in \mathscr{I}$, $m(I) < \delta$ and x is an endpoint of I.

Definition 2.2. A subset $\mathscr{C}$ of $\mathscr{I} \times [a, b]$ is said to be a *fine cover* of a set $X \subseteq [a, b]$ if for every $x \in X$ and every $\varepsilon > 0$ there is a pair $(I, x) \in \mathscr{C}$ with $m(I) < \varepsilon$ and x an endpoint of I.

Associated with the notions of a full cover and a fine cover is the construction of a pair of outer measures based on an arbitrary function defined on $\mathscr{I} \times [a, b]$. Suppose that h is a real-valued function whose domain includes at least one full cover of the interval $[a, b]$ (usually h will be defined on all of $\mathscr{I} \times [a, b]$) and let $\mathscr{C}$ be any subset of $\mathscr{I} \times [a, b]$ on which h is defined: then we compute

2.3. $$V(h, \mathscr{C}) = \sup \sum_{i=1}^{n} |h(I_i, x_i)|$$

where the supremum is with regard to all finite subsets $\{(I_i, x_i) : i = 1, 2, \ldots, n\}$ of $\mathscr{C}$ with I_i and I_j nonoverlapping for $i \neq j$.

From such a function h we then construct two outer measures on the interval $[a, b]$ as follows.

2.4. The set functions h^* and h_* are defined for subsets X of $[a, b]$ by

$$h^*(X) = \inf\{V(h, \mathscr{C}) : \mathscr{C} \text{ a full cover of } X\}$$

and

$$h_*(X) = \inf\{V(h, \mathscr{C}) : \mathscr{C} \text{ a fine cover of } X\}.$$

For the background to these concepts the reader is referred to [**4**] and [**5**]. In the particular case that $h(I, x) = m(I)$ the measures m^* and m_* coincide and yield the usual Lebesgue outer measure on $[a, b]$. One can even characterize the Lebesgue measurable sets in terms of full covers. For any real-valued function f on $[a, b]$ we shall agree to write Δf for the function

$$\Delta f([x, y], z) = \Delta f([x, y]) = f(y) - f(x)$$

defined on $\mathscr{I} \times [a, b]$. The measures Δf^* and Δf_* are referred to as the total variation measures associated with f and together will be useful in describing derivation properties of f.

The basic theorem that provides our starting point for investigating these measures is due to Henstock [**4**].

THEOREM 2.5. *Let h be an arbitrary real-valued function defined on some full cover of $[a, b]$. Then*

(i) *h^* and h_* are outer measures on $[a, b]$,*

(ii) *$h^* \geqslant h_*$,*

(iii) *every Borel set is h^*- and h_*-measurable,*

(iv) *for any increasing sequence* $\{X_n\}$ *of subsets of* $[a, b]$

$$h^*\left(\bigcup_{i=1}^{\infty} X_i\right) = \lim h^*(X_n).$$

Not only does this construction provide a link between derivation theory (that is the full and fine covers of a set which are a standard tool in derivation theory) and certain measures on the real line but there is also a direct link with integration theory. Indeed one may view the fundamental theorem of the calculus as arising from the fact that the same structures can be used to define derivatives and integrals. We outline briefly the integration theory that arises; as the properties of these integrals and their relationship to the Lebesgue and Denjoy-Perron integrals have been fully explored elsewhere we merely present the defintions appropriate to our context.

2.6. The Henstock–Kurzweil integral of a function f on $[a, b]$ is defined by writing

$$\int_{[x,y]} f(t)dt = F(y) - F(x)$$

if there is a function F on $[a, b]$ with $(\Delta F - fm)^*[a, b] = 0$, that is, H^* is the zero measure where $H(I, x) = \Delta F(I) - f(x)m(I)$. This integral has been shown to be equivalent to the Denjoy–Perron integral. It appears in **[4]** as the "Riemann-complete" integral and in [3; Appendix] as the "Perron–Ward" integral and a number of characterizations are possible.

2.7. For an arbitrary nonnegative function f defined on a subset X of $[a, b]$ we write

$$\overline{\int_X} f(t)dt = (fm)^*(X).$$

If f and g are finite nonnegative functions on X that agree almost everywhere then $\overline{\int_X} f(t)dt = \overline{\int_X} g(t)dt$. For this reason we may consider (2.7) as meaningful even if f is only defined almost everywhere in X.

It can be shown that in the event that f is nonnegative and measurable and X is a measurable subset of $[a, b]$ this is exactly the standard Lebesgue integral of f over X; if f is summable on $[a, b]$ then so are f^+ and f^- and

$$\int_{[a,b]} f(t)dt = \overline{\int_{[a,b]}} f^+(t)dt - \overline{\int_{[a,b]}} f^-(t)dt.$$

Finally we mention two further facts from the general theory that can be found in **[4]** and elsewhere.

LEMMA 2.8. *If $\mathscr{C}$ is a full cover of an interval $[\alpha, \beta]$ contained in $[a, b]$ then there is a partition of $[\alpha, \beta]$ from $\mathscr{C}$, that is a finite collection $\{(I_i, x_i) : i = 1, 2, \ldots, n\}$ with $\bigcup_{i=1}^{n} I_i = [\alpha, \beta]$ and I_i, I_j nonoverlapping for $i \neq j$.*

LEMMA 2.9. *If λ is a nonnegative subadditive function defined on $\mathscr{I}$ and $\lambda^*[a, b]$ is finite then there exists a nondecreasing function Λ on $[a, b]$ such that*

$$(\Delta\Lambda - \lambda)^*[a, b] = 0 .$$

Consequently $\lambda^* = \Delta\Lambda^*$ and $\lambda_* = \Delta\Lambda_*$.

3. *Properties of the total variation measures*

The measures Δf^* and Δf_* reflect the total variation of the function f and together provide a generalization of the usual Lebesgue–Stieltjes measures. Note that the constructions are based on the definitions of full and fine covers of a set and that these concepts are directly related to the ordinary derivative; if the center of attention were to be directed towards the symmetric derivative, the approximate derivative or some other generalized derivative corresponding changes would be made in definitions (2.1) and (2.2) and the measures would change accordingly. Thus we should consider the measures Δf^* and Δf_* here as generalizations of the Lebesgue–Stieltjes measures appropriate to the study of the ordinary derivative and the ordinary derived numbers of the function f.

Our main interest is in the situation in which Δf^* and Δf_* coincide as this has direct applications to the derivation theory of the function f. As an indication of the extreme case we point out that every function f in $C[a, b]$ excepting those in some first category subset of that space yields Δf_* as the zero measure and Δf^* as non σ-finite (see [**9**]).

THEOREM 3.1. *Suppose that $f \in C[a, b]$ is* VBG$_*$ *on a set $X \subseteq [a, b]$. Then Δf^* is σ-finite on X and $\Delta f^*(Y) = \Delta f_*(Y)$ for every $Y \subseteq X$.*

Proof. To begin with suppose that f has bounded variation on the whole interval $[a, b]$. Then certainly $\Delta f^* < +\infty$; we shall prove that Δf^* and Δf_* are identical. Without loss of generality (because of (2.9)) we may take f to be nondecreasing. In this case note that $\Delta f^*[\alpha, \beta] = f(\beta) - f(\alpha)$ because of (2.8).

Suppose that $\mathscr{C}$ is a fine cover of a set $Y \subseteq [a, b]$ and that μ_f^* is the usual Lebesgue–Stieltjes measure associated with f. The Vitali theorem applied to the measure μ_f^* shows the existence of a sequence $\{(I_i, x_i)\} \subseteq \mathscr{C}$ with the intervals I_i, I_j nonoverlapping for $i \neq j$ and with

$$\mu_f^*\left(Y \setminus \bigcup_{i=1}^{\infty} I_i\right) = 0 .$$

(For a proof of the Vitali theorem relative to a general measure of the type here see for example [**2**].)

For any interval (α, β) we have $\Delta f^*(\alpha, \beta) = f(\beta) - f(\alpha)$ and also $\mu_f^*(\alpha, \beta) = f(\beta) - f(\alpha)$ so that μ_f^* and Δf^* must agree at least on the Borel sets; in general then we have $\Delta f^* \leqslant \mu_f^*$ so that $\Delta f^*\left(Y \setminus \bigcup_{i=1}^{\infty} I_i\right) = 0$ too. We may now compute that

$$\Delta f^*(Y) \leqslant \Delta f^*(Y \setminus \bigcup I_i) + \sum \Delta f^*(I_i) = \sum \Delta f(I_i) \leqslant V(f, \mathscr{C}).$$

Since this holds for every fine cover $\mathscr{C}$ of Y it follows that $\Delta f^*(Y) \leqslant \Delta f_*(Y)$ and hence that $\Delta f^*(Y) = \Delta f_*(Y)$. As Y is arbitrary the theorem is proved under the additional assumption that f is of bounded variation.

Turning now to the general case we suppose that f is VBG$_*$ on a set X: then (since f is bounded) we may choose a sequence $\{F_n\}$ of closed sets with $F_1 \subseteq F_2 \subseteq \ldots, \bigcup F_n \supseteq X$ and f VB$_*$ on each set F_n. Let $\{I_i^{(n)}\}$ be the sequence of intervals contiguous to F_n in $[a, b]$: then f is VB on F_n and $\sum_{i=1}^{\infty} O(f, I_i^{(n)}) < +\infty$. We define functions g_n on $[a, b]$ with the following properties:

(i) g_n are continuous and of bounded variation on $[a, b]$;

(ii) $g_n(x) = f(x)$ for every $x \in F_n$;

(iii) $g_n(x) \leqslant f(x)$ everywhere; and

(iv) $\inf\{g_n(x) : x \in I_i^{(n)}\} = \inf\{f(x) : x \in I_i^{(n)}\}$

This is just a matter of extending g_n to each interval $I_i^{(n)}$; we shall not give the details. The following facts can now be checked.

(a) $(\Delta f - \Delta g_n)^*(F_n) = 0$.

(This is because $f(x) - g_n(x) = 0$ for every $x \in F_n$ and the series $\sum_{i=1}^{\infty} \sup\{|(\Delta f - \Delta g_n)[x, y]| : [x, y] \subseteq I_i^{(n)}\} \leqslant 2 \sum_{i=1}^{\infty} O(f, I_i^{(n)})$ converges.)

(b) $\Delta f^*(Y) = \Delta g_n^*(Y)$ and $\Delta f_*(Y) = \Delta g_{n*}(Y)$ for every $Y \subseteq F_n$.

Now combining (b) with the fact that for functions of bounded variation the theorem has been proved we obtain for sets $Y \subseteq X$,

$$\begin{aligned}
\Delta f^*(Y) &= \lim \Delta f^*(Y \cap F_n) && \text{(by (2.5 (iv)))} \\
&= \lim \Delta g_n^*(Y \cap F_n) && \text{(by (b))} \\
&= \lim \Delta g_{n*}(Y \cap F_n) && \\
&= \lim \Delta f_*(Y \cap F_n) && \text{again by (b))} \\
&\leqslant \Delta f_*(Y).
\end{aligned}$$

This proves that Δf^* and Δf_* agree on subsets of X and even expresses $X = \bigcup (X \cap F_n)$ as a union of sets of finite Δf^* measure, completing the proof of the theorem.

There are some partial converses available for Theorem (3.1). The σ-finiteness of the measure Δf^* is largely a characterization of the VBG_* property. In particular we can prove the following.

3.2. *If f is a real-valued function on $[a, b]$ and Δf^* is σ-finite on X then f is* VBG_* *on X.*

The proof is almost identical with that of a similar result in Henstock [5; pp. 8–9]. It seems likely that the agreement of Δf^* and Δf_* on a set X (with $f \in C[a, b]$) requires f to be VBG_*, but we do not have a proof. It is not difficult to establish this if f is assumed also to fulfill Lusin's condition N on the set X, but we must leave the problem unresolved in general.

We turn now to considerations related to absolute continuity.

THEOREM 3.3. *Suppose that $f \in C[a, b]$. Then $\Delta f_*(N) = 0$ whenever $m(f(N)) = 0$, and $m(f(N)) = 0$ whenever $\Delta f^*(N) = 0$. In particular if f is also* VBG_* *on a set N then $\Delta f^*(N)$, $\Delta f_*(N)$ and $m(f(N))$ must all vanish together.*

Proof. The first part is proved in [9] and the second follows easily from a well-known lemma of Saks and Sierpinski [8; p. 211]. Write, for any function f on $[a, b]$, $(m \circ f)(I, x) = m(f(I))$; then this lemma gives in general the estimate

$$m(f(E)) \leqslant 2(m \circ f)_*(E) \quad (E \subseteq [a, b]).$$

Since for continuous f we must have

$$(m \circ f)_* \leqslant (m \circ f)^* \leqslant 2\Delta f^*,$$

it follows that $m(f(N)) \leqslant 4\Delta f^*(N)$ and so if $\Delta f^*(N) = 0$ then certainly $m(f(N)) = 0$ as required.

We conclude this section with some simple applications of these ideas to obtain characterizations of the class of functions that are VBG_* or ACG_* on a set. The first is essentially the theorem of Banach–Zarecki in view of Theorem (3.3).

THEOREM 3.4. *Suppose that $f \in C[a, b]$. Then f is* AC_* *[respectively* ACG_**] on a set $X \subseteq [a, b]$ if and only if f is* VB_* *[respectively* VBG_**] on X and $\Delta f^*(N) = 0$ for every set N of measure zero, $N \subseteq X$.*

Proof. If f has bounded variation on $[a, b]$ and F is the total variation of f then it is well known that f is absolutely continuous on $[a, b]$ if and only if μ_F vanishes on null sets where μ_F is the Lebesgue–Stieltjes measure associated with F (see for example [7; p. 249]). Since μ_F and Δf^* agree at least on Borel sets this statement applies equally to Δf^*.

More generally if f is VBG_* on a set X we will construct as in the proof of (3.1) sequences $\{F_n\}$, $\{I_i^{(n)}\}$, $\{g_n\}$; but here we will insist that the functions g_n be even absolutely continuous on each interval $I_i^{(n)}$. Then as before for any set $N \subseteq X$, we have

$$\Delta f^*(N) = \lim \Delta f^*(N \cap F_n) = \lim \Delta g_n{}^*(N \cap F_n).$$

Clearly $\Delta f^*(N) = 0$ for a set $N \subseteq X$ if and only if $\Delta g_n^*(N \cap F_n) = 0$ for every index n. If f is ACG$_*$ then we can arrange to choose the sets F_n so that f is AC on each F_n. But then g_n will be absolutely continuous on $[a, b]$: this means that any null set $N \subseteq X$ will have $\Delta g_n^*(N) = 0$ and consequently $\Delta f^*(N) = 0$. On the other hand if $\Delta f^*(N) = 0$ for every null set $N \subseteq X$ it will follow that g_n is absolutely continuous on $[a, b]$ for each index n. To see this consider any null set $M \subseteq [a, b]$: then

$$\Delta g_n^*(M) \leqslant \Delta g_n^*(M \cap F_n) + \sum \Delta g_n^*(M \cap I_i^{(n)}) = 0$$

since g_n is absolutely continuous on each interval $I_i^{(n)}$ and since $\Delta g_n^*(M \cap F_n) = \Delta f^*(M \cap F_n) = 0$ by hypothesis. But if g_n is absolutely continuous then f is AC$_*$ on F_n and this expresses f as ACG$_*$ on X.

THEOREM 3.5. *Let $f \in C[a, b]$ be* VBG$_*$ *on a set X. If f has at least one finite derived number at every point of X with at most countably many exceptions then f is* ACG$_*$ *on X.*

Proof. This should be compared with a similar result of Denjoy [**8**; p. 235]. Because of (3.4) we need only show that $\Delta f_*(N) = 0$ for every set $N \subseteq X$ of measure zero. Since f is continuous Δf_* will vanish on countable sets so that we may consider f to have a finite derived number everywhere in X. Let $g(x) \in \operatorname{der} f(x)$ for every $x \in X$. Then considering fine covers of X we have immediately $(\Delta f - gm)_*(X) = 0$, and so $\Delta f_*(Y) \leqslant (gm)^*(Y)$ for all $Y \subseteq X$. If $m(N) = 0$ then it is easy to see that $(gm)^*(N) = 0$ and so $\Delta f_*(N) = 0$ whenever $N \subseteq X$ and $m(N) = 0$.

4. *Existence of the derivative*

If f has g as a derivative everywhere in a set X then it is clear from a consideration of appropriate full covers of X that

$$(\Delta f - gm)^*(X) = 0 .$$

This provides an immediate connection between Δf^* and $(gm)^*$. Similarly if f has a derived number $g(x)$ at every point of a set X ($g(x)$ finite) then a consideration of appropriate fine covers of X gives

$$(\Delta f - gm)_*(X) = 0 .$$

This then provides a relation between Δf_* and $(gm)^*$. These observations together with the properties of the two measures provide a method that is useful in determining properties of the function f from information about its derived numbers or in determining properties of the derived numbers from information about the function. The remainder of the paper is devoted to exploiting these methods. Most of the results are not new but much is revealed by placing them within this unifying point of view.

THEOREM 4.1 [Denjoy–Lusin]. *Suppose that $f \in C[a, b]$ is* VBG$_*$ *on a set $X \subseteq [a, b]$. Then the set of points of X at which f has no finite derivative is of measure* *zero and the set of points of X at which f has no finite or infinite derivative is of Δf^*-measure zero.*

Proof. The standard proof is in Saks [8; Theorem 7.2, p. 230]. Our proof arises entirely from the equality $\Delta f^* = \Delta f_*$ on X. Let X_r denote the set of points x in X for which $r \in \operatorname{der}^+ f(x)$: then, as mentioned at the beginning of the section, $(\Delta f - rm)_*(X_r) = 0$. Consequently for subsets $Y \subseteq X_r$ we have

$$\Delta f_*(Y) \leqslant |r| m^*(Y) = |r| m_*(Y) \leqslant \Delta f^*(Y)$$

and so because of the equality of Δf^* and Δf_* on X we have $\Delta f^*(Y) = |r| m^*(Y)$ for $Y \subseteq X_r$. In particular if $|r| \neq |s|$ then $m^*(X_r \cap X_s) = \Delta f^*(X_r \cap X_s) = 0$.

Let X_1 be the set of points in X at which $D_+ f(x) \neq D^+ f(x)$. Then, since f is continuous, the interval $[D_+ f(x), D^+ f(x)] = \operatorname{der}^+ f(x)$ is nondegenerate for every $x \in X_1$ and so X_1 can be written as a countable union of sets that are of m and Δf^* measure zero, namely

$$X_1 = \bigcup \{X_r \cap X_s : r, s \text{ rational}, |r| \neq |s|\} .$$

Similarly the set of points $X_2 \subseteq X$ at which $D_- f(x) \neq D^- f(x)$ has $m(X_2) = \Delta f^*(X_2) = 0$. Finally by a Theorem of G. C. Young (see [1; p. 63]) the set of points $X_3 \subseteq X$ at which $D^+ f(x) < D_- f(x)$ is denumerable so that certainly $m(X_3) = \Delta f^*(X_3) = 0$ too.

Thus the derivative exists finitely or infinitely everywhere in $X \setminus (X_1 \cup X_2 \cup X_3)$ and the first part of the theorem is proved. For the second part it is straightforward to show that the σ-finiteness of Δf^* on X forces the sets $\{x \in X : D^+ f(x) = +\infty\}$, $\{x \in X : D^- f(x) = +\infty\}$ etc. to have m-measure zero. In fact, however, the set of points on which any function has an infinite derivative is necessarily of measure zero and the final assertion then necessarily follows from the first.

COROLLARY 4.2. *Suppose that $f \in C[a, b]$ is VBG_* on a set $X \subseteq [a, b]$. Then a necessary and sufficient condition for f to be ACG_* on X is that $m(f(N)) = 0$ where $N = \{x \in X : f'(x) = \pm\infty\}$.*

Proof. Since N is necessarily of measure zero the condition is certainly necessary. To prove that it is sufficient let $Y = \{x \in X : -\infty < f'(x) < +\infty\}$ and $Z = \{x \in X : f'(x) \text{ does not exist}\}$. By the theorem $\Delta f^*(Z) = 0$ and our methods give for subsets $Y' \subseteq Y$ that $\Delta f^*(Y') = (f'm)^*(Y')$. Thus if $M \subseteq X$ has measure zero, $(f'm)^*(M \cap Y) = 0$ and we may write

$$\Delta f^*(M) \leqslant \Delta f^*(M \cap Y) + \Delta f^*(M \cap Z) + \Delta f^*(M \cap N)$$

$$\leqslant (f'm)^*(M \cap Y) + \Delta f^*(M \cap N) \leqslant \Delta f^*(N) .$$

Thus the vanishing of $\Delta f^*(N)$ implies the vanishing of $\Delta f^*(M)$ for all subsets M of X of measure zero. The corollary now follows from (3.3) and (3.2).

5. *Integral of the derivative*

The connection between the derivative and the integral is contained in the assertion that $(\Delta f - f'm)^*(X) = 0$, and from this we can easily obtain results. We list the most obvious consequences.

5.1. *If f' exists everywhere in $X \subseteq [a, b]$ then*

$$\overline{\int_X} |f'(x)|dx = \Delta f^*(X).$$

5.2. *If f' exists everywhere in a set X except possibly on a subset N, where $f \in C[a, b]$, then*

$$\overline{\int_X} |f'(x)|dx = \Delta f^*(X)$$

provided that

(i) *N is countable, or*

(ii) *$m(N) = 0$ and f is* ACG$_*$ *on X, or*

(iii) *$m(N) = m(f(N)) = 0$ and f is* VBG$_*$ *on X.*

5.3. *If g is a finite function on $[a, b]$, $f \in C[a, b]$, and $f'(x) = g(x)$ everywhere in $[a, b]$ with at most countably many exceptions, then*

$$\int_{[x,y]} g(t)dt = f(y) - f(x) \quad \textit{for} \quad [x, y] \subseteq [a, b].$$

5.4. *If g is a finite function on $[a, b]$, f is* ACG$_*$ *on $[a, b]$ and $f' = g$* a.e. *in $[a, b]$ then*

$$\int_{[x,y]} g(t)dt = f(y) - f(x) \quad \textit{for} \quad [x, y] \subseteq [a, b].$$

All four of these are relatively trivial consequences of the relation $(\Delta f - f'm)^*(X) = 0$, which holds if f' exists finitely on a set X, together with the results of Section 3 above. Similarly if one has a finite function g on a set X with $g(x) \in \operatorname{der} f(x)$ for every $x \in X$ the relation $(\Delta f - gm)_*(X) = 0$ holds and one can obtain a number of easy consequences.

5.5. *If f has a finite derived number $g(x)$ everywhere in a set X then*

$$\Delta f_*(X) \leqslant \overline{\int_X} |g(x)|dx.$$

5.6. *If* $f \in C[a,b]$ *is* VBG$_*$ *on a set* X *and* f *has a finite derived number* $g(x)$ *everywhere in* X *then*

$$\Delta f^*(X) \leqslant \overline{\int_X} |g(x)|\,dx\,.$$

6. *Vanishing derivatives*

There are a number of theorems which assert that a function is constant on the basis of a vanishing derivative; in essence these are all generalizations of the result of elementary calculus that only constant functions have everywhere vanishing derivatives. As soon as an exceptional set is permitted some regularity condition must be imposed. In this section we list some of the consequences of the following observations:

(a) f is constant on $[a,b]$ if and only if $\Delta f^*[a,b] = 0$;

(b) if $f' = 0$ on a set X then $\Delta f^*(X) = 0$;

(c) if $0 \in \operatorname{der} f(x)$ everywhere in a set X then $\Delta f_*(X) = 0$.

Assertion (a) follows directly from Lemma 2.8 and (b) and (c) follow from (5.1) and (5.5). Most of the consequences are easy to prove using these three facts.

6.1. *Let* $f \in C[a,b]$ *have a zero derivative everywhere on* $[a,b]$ *except possibly in a set* N. *Then if any one of the following are true* f *must be constant on* $[a,b]$:

(i) N *is countable*;

(ii) $\Delta f^*(N) = 0$;

(iii) f *is* VBG$_*$ *and* $m(f(N)) = 0$;

(iv) f *is* ACG$_*$ *and* $m(N) = 0$.

To prove this is merely to prove that, in each of the four cases, $\Delta f^*(N) = 0$ and then apply (a) and (b). For the derived numbers we have the following.

6.2. *Suppose that* $f \in C[a,b]$ *is* VBG$_*$ *on* $[a,b]$ *and that everywhere in* $[a,b]$ *with at most countably many exceptions* f *has a derived number equal to zero. Then* f *is constant.*

Compare this with Saks [**6**; Theorem 4.7, p. 272] which asks instead for one of the four Dini derivatives to vanish in which case f would have to be VBG$_*$ and (6.2) could be used. The proof here just uses (c) and the fact that $\Delta f^* = \Delta f_*$.

Many more results of this nature are possible. We give one more sample: a result of Leonard (cf. [**1**; p. 192]) can be used if one knows that the upper Dini derivative $D^+f(x) = 0$ a.e. and that it is in Baire class 1. Here we add in the VBG$_*$ assumption but then we can work with the derived numbers rather than an extreme derivative.

6.3. *Suppose that* $f \in C[a, b]$ *is* VBG$_*$ *on* $[a, b]$ *and that there exists a function* g *(possibly infinite valued) such that*

(i) g *is in Baire class* 1,

(ii) $g(x) \in \operatorname{der}^+ f(x)$ *for every* $x \in [a, b]$,

(iii) $g = 0$ a.e. *in* $[a, b]$.

Then f *is constant on* $[a, b]$.

Assertion (ii) may be weakened to require only that $g(x) \in \operatorname{der} f(x)$ everywhere provided we also assume that g is at least Darboux continuous. We prove this by establishing that $\Delta f_*[a, b] = 0$ and then, since f is VBG$_*$, it follows that $\Delta f^*[a, b] = 0$ and hence that f is constant. Let G be the set of points x in $[a, b]$ such that Δf_* vanishes on some neighbourhood of x. Then clearly G is open and $\Delta f_*(G) = 0$. We wish to prove that $G = [a, b]$; write $E = [a, b] \setminus G$ and note that E is perfect and nowhere dense. The former is obvious; to see that E is nowhere dense consider that it contains instead an interval $[\alpha, \beta]$. Since g is of Baire class 1 it must be continuous at a dense subset D of $[\alpha, \beta]$. In particular we must have for some subinterval $[\alpha', \beta']$ of $[\alpha, \beta]$ that g is finite on $[\alpha', \beta']$ and vanishes a.e. Consequently

$$\Delta f_*[\alpha, \beta'] \leqslant (gm)^*([\alpha', \beta']) = 0$$

and this contradicts the definition of E.

Suppose now that $E \neq \emptyset$ and let $\{(\alpha_k, \beta_k)\}$ be the component intervals of G. Since E is nowhere dense the sequence $\{\alpha_k\}$ must be dense in E. But $g(\alpha_k) = 0$ for each index because f is constant on $[\alpha_k, \beta_k]$, so that $\operatorname{der}^+ f(\alpha_k) = \{0\}$. Because g is in Baire class 1 its restriction to E must be continuous at some point $x_0 \in E$ and this means that $g(x_0)$ can only be zero. In particular there is an interval (α, β) such that $x_0 \in (\alpha, \beta) \cap E$ and g is finite on $(\alpha, \beta) \cap E$, with $g = 0$ a.e. there. From this we obtain that

$$\Delta f_*((\alpha, \beta) \cap E) \leqslant (gm)^*((\alpha, \beta) \cap E) = 0,$$

and hence that

$$\Delta f_*(\alpha, \beta) \leqslant \Delta f_*((\alpha, \beta) \cap G) + \Delta f_*((\alpha, \beta) \cap E) = 0,$$

so that $x_0 \in G$ which contradicts $x_0 \in E$. This contradiction proves that $E = \emptyset$ and proves the result.

7. *Monotonicity theorems*

These methods apply without serious modifications to the study of conditions that ensure that a given function is monotonic. For any function f on $[a, b]$ we define

$$\Delta f^+(I) = \max\{\Delta f(I), 0\},$$

and

$$\Delta f^-(I) = \max\{-\Delta f(I), 0\}.$$

Then the measures $(\Delta f^+)^*$ and $(\Delta f^-)^*$ carry the information needed: the function f is nondecreasing on $[a, b]$ if and only if $(\Delta f^-)^*[a, b] = 0$. This is an immediate consequence of Lemma (2.8). The method then for determining that f is nondecreasing is to use information about the derived numbers to conclude that $(\Delta f^-)^*$ or $(\Delta f^-)_*$ vanishes on certain sets.

We begin with the analogue of Theorem (3.1).

THEOREM 7.1. *Let $f \in C[a, b]$ be* VBG_* *on a set $X \subseteq [a, b]$. Then $(\Delta f^+)^* = (\Delta f^+)_*$ and $(\Delta f^-)^* = (\Delta f^-)_*$ on X and both are σ-finite on X.*

Proof. Suppose to begin with that f is even of bounded variation on $[a, b]$. Then Δf^+ is nonnegative, subadditive, and $(\Delta f^+)^*$ is finite (since it bounded by $V(f, [a, b])$). By (2.9) then, there is a nondecreasing function g in $C[a, b]$ such that

$$(\Delta f^+ - \Delta g)^*[a, b] = 0.$$

From this we see that $(\Delta f^+)^* = \Delta g^*$ and $(\Delta f^+)_* = \Delta g_*$ and Theorem 3.1 applies to show that $\Delta g^* = \Delta g_*$ which gives our result. The same argument works for Δf^-.

The extension to the situation in which f is VBG_* on a set X can be made by repeating the construction used in (3.1) and so reducing this back to the case of bounded variation. The details are almost identical and will not be repeated.

The connection of the derivative and the derived numbers to these measures is straightforward; the proof requires only a consideration of appropriate full and fine covers of the set X.

LEMMA 7.2. *If f has a nonnegative derivative everywhere in a set X then $(\Delta f^-)^*(X) = 0$. If f has a nonnegative derived number everywhere in a set X then $(\Delta f^-)_*(X) = 0$.*

Proof. For any $\varepsilon > 0$ let $\mathscr{C}$ be the collection of pairs $([x, y], x)$ or $([y, x], x)$ such that $[f(y) - f(x)]/(y - x) \geqslant -\varepsilon$. Then $\mathscr{C}$ is a full cover of X and $V(\Delta f^-, \mathscr{C}) \leqslant \varepsilon(b - a)$. From this we obtain that $(\Delta f^-)^*(X) = 0$, since ε is arbitrary. The second assertion of the lemma is proved similarly using fine covers.

We now list some of the consequences of the lemma. The proofs in each case are simply a matter of using the properties of the measures to obtain that $(\Delta f^-)^*$ vanishes on $[a, b]$, which means that f is nondecreasing on that interval.

7.3. *Let $f \in C[a, b]$ have a nonnegative derivative everywhere on $[a, b]$ except possibly in a set N. Then if any one of the following is true f must be nondecreasing on $[a, b]$:*

(i) *N is countable;*

(ii) *$\Delta f^*(N) = 0$ (or even $(\Delta f^-)^*(N) = 0$);*

(iii) *f is* VBG_* *and $m(f(N)) = 0$;*

(iv) *f is* ACG_* *and $m(N) = 0$.*

For the derived numbers as well we have several results.

7.4. *Let* $f \in C[a, b]$ *be* VBG$_*$ *on* $[a, b]$ *and have everywhere at least one nonnegative derived number except possibly at the points of a set* N *with* $m(f(N)) = 0$. *Then* f *is nondecreasing on* $[a, b]$.

If $X = [a, b] \setminus N$ then (7.2) gives that $(\Delta f^-)_*(X) = 0$ and (3.3) shows that $(\Delta f^-)_*(N) \leqslant \Delta f^*(N) = m(f(N)) = 0$. Thus $(\Delta f^-)_*([a, b]) = 0$ and then (7.1) proves that $(\Delta f^-)^*[a, b] = 0$ as required.

Finally we mention a result similar to that of Leonard [**1**; p. 192]; the proof follows the same lines as that for (6.3) above.

7.5. *Suppose that* $f \in C[a, b]$ *is* VBG$_*$ *on* $[a, b]$ *and that there exists a function* g *(possibly infinite valued) such that*

(i) g *is in Baire class* 1,

(ii) $g(x) \in \operatorname{der}^+ f(x)$ *for every* $x \in [a, b]$,

(iii) $g \geqslant 0$ a.e. *in* $[a, b]$.

Then f *is nondecreasing on* $[a, b]$.

Again (ii) may be weakened to require merely that $g(x) \in \operatorname{der} f(x)$ everywhere provided g is then assumed to be Darboux continuous.

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Mathematics Department,
Simon Fraser University,
B.C.,
Canada V5A 1S6.