SOME THEOREMS FOR EXTREME DERIVATES

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1. *Introduction*

Our attention in this article is focused on two classical theorems for the Dini derivatives. The first of these was proved by W. H. Young [8] in 1908 and the second, closely related result, was obtained several years later by Sierpiński [6] and G. C. Young [7] independently. Both theorems have been subjected to extensive generalizations.

THEOREM I (W. H. Young). *Let F be a continuous real-valued function defined on the real line. Then the set of points x at which either $D^+F(x) \neq D^-F(x)$ or $D_+F(x) \neq D_-F(x)$ is of the first category.*

THEOREM II (G. C. Young and W. Sierpiński). *Let F be an arbitrary real-valued function defined on the real line. Then the set of points x at which either $D^+F(x) < D_-F(x)$ or $D_+F(x) > D^-F(x)$ is denumerable.*

The theorems are related technically as well as historically. In order to obtain a proof one can use a familiar decomposition device: if everywhere in a set X the inequality $D^+F(x) < \alpha$ (respectively $D_+F(x) > \alpha$) is maintained then there is a sequence of sets $\{X_n\}$ covering X on each of which the function $F(x) - \alpha x$ (respectively $\alpha x - F(x)$) is decreasing. A similar decomposition holds as well for the left Dini derivative. If the exceptional sets of the two theorems are decomposed in the correct manner using this device one can then verify (in the first case) that each member of the decomposition is nowhere dense or (in the second case) that each member is finite.

There have been numerous generalizations of these theorems. Certainly if a generalized derivation permits such a decomposition the analogous theorems hold with essentially the same proof: in this way it can be shown that the (bilateral) approximate derivates, the preponderant derivates, the selective derivates, the qualitative derivates, and several others can all replace the right or left Dini derivates in these theorems. The approximate Dini derivatives do not, however, allow such a decomposition and it is not immediately clear that these theorems extend to them. In fact it is only recently that appropriate versions have appeared. Pu, Chen and Pu [5] in 1974 showed that Theorem I remains true if the approximate Dini derivatives are substituted in that statement; Zajíček [9] in 1973 showed that, in contrast, Theorem II cannot be so altered even if F is assumed to be continuous. He then showed that a weaker version is available.

The purpose of this paper is to present a perspective on these theorems that helps to explain how the original theorems arise and how they can be extended to a variety of generalized derivatives. In particular we give abstract versions of the two theorems of Pu, Chen and Pu and Zajíček mentioned above. Our formulation appears in the

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setting of path derivates introduced in Bruckner, O'Malley and Thomson [3]. In contrast the reader might compare the cluster set viewpoint pursued by Bruckner and Goffman [2] as well as a number of later authors.

2. *Preliminaries*

We shall reproduce here the definitions needed for a discussion of these problems in the setting of path derivates. A more elaborate account appears in [3].

DEFINITION 2.1. By a *path leading to* x we mean merely a set E_x of real numbers, containing x and having x as a point of accumulation. A system

$$E = \{E_x : x \in \mathbb{R}\}$$

is said to be a system of paths if each E_x is a path leading to x.

DEFINITION 2.2. Let F be a real-valued function defined on the real line and let $E = \{E_x : x \in \mathbb{R}\}$ be a system of paths. Then the E-derivates of F are defined as

$$\bar{F}'_E(x) = \limsup_{\substack{y \to x \\ y \in E_x}} \frac{F(y)-F(x)}{y-x} \quad \text{and} \quad \underline{F}'_E(x) = \liminf_{\substack{y \to x \\ y \in E_x}} \frac{F(y)-F(x)}{y-x}.$$

With these definitions to hand we may express the program of this paper as an investigation of the conditions that a system of paths $E = \{E_x : x \in \mathbb{R}\}$ should meet in order that appropriate generalizations of Theorems I and II can be made for the derivates $\bar{F}'_E$ and $\underline{F}'_E$. Without any further conditions on the nature of the paths nothing much can be said about the behaviour of the derivates (see [3, Theorem 3.1]). The type of assumption that has proven useful in these investigations involves the manner in which pairs of paths E_x and E_y intersect. Here we indicate two such assumptions; more have appeared in [3].

DEFINITION 2.3. Let $E = \{E_x : x \in \mathbb{R}\}$ be a system of paths.

(2.3.1) E is said to satisfy the *intersection condition* if there is a positive function δ on $\mathbb{R}$ such that whenever $0 < y-x < \min\{\delta(x), \delta(y)\}$ then $E_x \cap E_y \cap [x, y] \neq \emptyset$.

(2.3.2) E is said to satisfy the *one-sided sharp external intersection condition* if for every $\varepsilon > 0$ there is a positive function δ such that whenever $0 < y-x < \min\{\delta(x), \delta(y)\}$ then one at least of the intersections

$$E_x \cap E_y \cap (x-\varepsilon(y-x), x] \quad \text{and} \quad E_x \cap E_y \cap [y, y+\varepsilon(y-x))$$

is non-empty.

Note that (2.3.2) is a sharper version of the external intersection condition given in [3]; this explains the name.

A number of generalized derivates can be expressed as path derivates relative to systems that satisfy the intersection condition. The approximate Dini derivatives do not allow such an expression but they can be thought of as path derivates relative to systems that have the weaker property (2.3.2). This is the content of our first lemma.

LEMMA 2.4. *Let $E = \{E_x : x \in \mathbb{R}\}$ be a system of paths such that each E_x has right (left) inner density 1 at x. Then E satisfies the one-sided sharp external intersection condition.*

Proof. We use $|A|_*$ to denote the inner measure of the set A. Given any $0 < \varepsilon < 1$ choose $\eta(x) > 0$ such that if $0 < t < \eta(x)$ then $|E_x \cap (x, x+t)|_* > t(1-\frac{1}{4}\varepsilon)$, and define $\delta(x) = \eta(x)/(1+\varepsilon)$. Now if $0 < y-x < \min\{\delta(x), \delta(y)\}$ and $z = y+\varepsilon(y-x)$ then

$$0 < z-x = (1+\varepsilon)(y-x) < (1+\varepsilon)\delta(x) = \eta(x),$$

so that $|E_x \cap (x, z)|_* > (1-\frac{1}{4}\varepsilon)(z-x) = (1-\frac{1}{4}\varepsilon)(1+\varepsilon)(y-x)$. This gives

$$|E_x \cap (y, z)|_* > [(1-\tfrac{1}{4}\varepsilon)(1+\varepsilon)-1](y-x) > (y-z)/2 .$$

But we also have

$$|E_y \cap (y, z)|_* > (1-\tfrac{1}{4}\varepsilon)(y-z) > (y-z)/2 .$$

It follows now that the intersection $E_x \cap E_y \cap (y, z)$ cannot be empty and so we have verified that assertion (2.3.2) holds, and the lemma is proved.

We conclude this section by presenting a definition of the only other concept needed in order to express our results. This is the notion of set porosity due to Dolženko [4]. If $A \subset \mathbb{R}$ and $x \in \mathbb{R}$ then the *right hand porosity* of A at x is the number

$$\limsup_{h \to 0+} l(A, x, h)/h$$

where $l(A, x, h)$ denotes the length of the largest open subinterval of $(x, x+h)$ that contains no point of A. Left hand porosity is similarly defined. A set A is said to be *strongly porous* on the right (left) at a point if the corresponding porosity is 1. Note that positive outer density on the right (or left) at a point implies that a set is not strongly porous on that side.

Finally we might remark on the way in which these path derivatives and path derivates were intended for use. In any given setting in which a generalized derivation process has been defined the paths $\{E_x\}$ are chosen tailored to the function F whose derivates are under investigation, and then perhaps altered. Thus some derivations are *a priori* defined as a type of path derivate, while others require some manipulation. Since most applications of extreme derivates use only inequalities it is enough that an extreme derivate $\bar{F}'_E$ should approximate the generalized one. For example if $D^+_{\text{ap}}F(x) = c$ is the upper approximate Dini derivative of F at a point x then for any $c' > c$ the set $\left\{y : \dfrac{F(y)-F(x)}{y-x} > c'\right\}$ has outer density 0 on the right at x, and hence the set $E_x = \left\{y : \dfrac{F(y)-F(x)}{y-x} \leqslant c'\right\}$ has inner density 1 on the right at x and $c \leqslant \bar{F}'_E(x) \leqslant c'$. Trivially, then, any theorem we prove below for path derivates of this type can be extended by approximation arguments to the approximate Dini derivates. Similarly Theorems 3.1, 3.2, 4.1 and 4.2 apply to any derivation process expressible directly as path derivates possessing the appropriate properties or to any derivation process that can be so approximated.

3. *The W. H. Young relations*

Theorem I was expressed by Young as asserting that 'there is no distinction of right and left with respect to derivates except at points of a set of the first category' [**8**, p. 306]. The real nature of this relationship was clarified by Bruckner and Goffman [**2**] who showed that in fact for any continuous function F,

$$\bar{F}^*(x) = D^+F(x) = D^-F(x) = \&\text{c.} \qquad \text{and} \qquad \underline{F}^*(x) = D_+F(x) = D_-F(x) = \&\text{c.}$$

must hold residually, where by $\bar{F}^*$ and $\underline{F}^*$ we mean the *strong* extreme derivates of F defined as

$$\bar{F}^*(x) = \limsup_{\substack{(y,z)\to(x,x)\\ y\neq z}} \frac{F(y)-F(z)}{y-z} \qquad \text{and} \qquad \underline{F}^*(x) = \liminf_{\substack{(y,z)\to(x,x)\\ y\neq z}} \frac{F(y)-F(z)}{y-z},$$

and where the '&c.' involved any choice of directional approach (see [**2**] or [**1**, pp. 67–70] for details). Because of this theorem it seems more appropriate to interpret Theorem I as asserting the equality of certain extreme derivates with the corresponding strong extreme derivates except on a first category set, and we shall call any such assertion a 'W. H. Young relation'.

The problem we shall address in this section is to determine conditions that a system of paths $E = \{E_x : x \in \mathbb{R}\}$ should satisfy in order that $\bar{F}'_E(x) = \bar{F}^*(x)$ and $\underline{F}'_E(x) = \underline{F}^*(x)$ should hold residually for any continuous function F. Our first theorem is in essence [**3**, Theorem 4.3] and merely uses the fact that for path derivates relative to a system that satisfies an intersection condition a decomposition of the type discussed in the introduction is available. This includes as a special case the original Theorem I as well as versions for the approximate (bilateral) derivates and a number of others, but does not include the theorem of Pu, Chen and Pu [**5**] for the approximate Dini derivatives.

THEOREM 3.1. *Let F be a continuous function and let E be a system of paths that satisfies the intersection condition. Then the set of points x at which either $\bar{F}'_E(x) \neq \bar{F}^*(x)$ or $\underline{F}'_E(x) \neq \underline{F}^*(x)$ is of the first category.*

Proof. See [**3**, Theorem 4.3].

In order to obtain a general theorem that is applicable as a special case to the approximate Dini derivatives we need only ask of the paths that they not be too thin. Quite surprisingly only a very weak porosity requirement is enough to ensure that a W. H. Young relation holds.

THEOREM 3.2. *Let F be a continuous function and let $E = \{E_x : x \in \mathbb{R}\}$ be a system of paths such that at each x the path E_x is not strongly porous on at least one side. Then the set of points x at which either $\bar{F}'_E(x) \neq \bar{F}^*(x)$ or $\underline{F}'_E(x) \neq \underline{F}^*(x)$ is of the first category.*

Proof. Let p be a rational number in the interval $(0, 1)$ and let X_p (respectively Y_p) denote the set of points x such that the right hand (respectively left hand) porosity of E_x at x is less than p. We shall show that the sets

$$\hat{X}_p = \{x \in X_p : \bar{F}'_E(x) \neq \bar{F}^*(x)\} \qquad \text{and} \qquad \hat{Y}_p = \{x \in Y_p : \bar{F}'_E(x) \neq \bar{F}^*(x)\}$$

are of the first category. From this it will follow that the set

$$\bigcup\{\hat{X}_p \cup \hat{Y}_p : p \in (0,1),\ p \text{ rational}\},$$

which includes every point at which $\bar{F}'_E(x) \neq \bar{F}^*(x)$, is also first category. Since the lower extreme derivates can be handled in the same fashion the theorem will be proved.

For rational numbers α and β define the sets

$$\hat{X}_{p\alpha\beta} = \{x \in \hat{X}_p : \bar{F}'_E(x) < \alpha < \beta < \bar{F}^*(x)\}$$

and choose positive functions δ_1 and δ_2 on $\hat{X}_{p\alpha\beta}$ such that if $x \in \hat{X}_{p\alpha\beta}$ and $0 < h < \delta_1(x)$ then $l(E_x, x, h) < ph$ and if $y \in E_x$ and $0 < y-x < \delta_2(x)$ then $F(y)-F(x) < \alpha(y-x)$. Define $\delta = \min\{\delta_1, \delta_2\}$ and let $\{\hat{X}^n_{p\alpha\beta}\}_{n=1}^{\infty}$ be a δ-decomposition of the set $\hat{X}_{p\alpha\beta}$ (see [3, Definition 3.7]), that is, just a relabelling of the double sequence

$$\{x \in \hat{X}_{p\alpha\beta} : \delta(x) > 1/m\} \cap \left[\frac{j}{m}, \frac{j+1}{m}\right]$$

for $m = 1, 2, 3, \ldots$ and $j = 0, \pm 1, \pm 2, \ldots$.

We claim that each set $\hat{X}^n_{p\alpha\beta}$ is nowhere dense; since the set $\hat{X}_p$ is a countable union of such sets (for $n = 1, 2, 3, \ldots$ and for α and β rational) it will follow that $\hat{X}_p$ is first category as required. The argument for $\hat{Y}_p$ is similar. To obtain a contradiction let us suppose that some set $\hat{X}^n_{p\alpha\beta}$ is dense in an interval (c, d); the inequality

$$\text{(3.2.1)} \qquad \frac{F(y)-F(z)}{y-z} \leqslant \beta \qquad \text{for } y, z \in (c, d)$$

must hold. We show this by an indirect argument. If (3.2.1) fails then there are points $c < x_1 < y_1 < d$ with $G(x_1) < G(y_1)$ where G is the function $G(x) = F(x)-\beta x$. Choose the point x_0 as $x_0 = \sup\{x \in [x_1, y_1] : G(x) \leqslant G(x_1)\}$. Certainly $x_1 \leqslant x_0 < y_1$ so that $h = y_1 - x_0$ is positive. As we have assumed that the set $\hat{X}^n_{p\alpha\beta}$ is dense in the interval (c, d) and G is continuous there we may select a point x in $\hat{X}^n_{p\alpha\beta}$ such that $0 < x_0 - x < (1-p)h$ and $G(x) < G(x_0)+(\beta-\alpha)(1-p)h$. By the nature of the decomposition we have $0 < h < \delta_1(x)$ and $0 < h < \delta_2(x)$ so that $l(E_x, x, h) < ph$, and hence there is a point z in $E_x \cap (x+(1-p)h, x+h)$ with $F(z)-F(x) < \alpha(z-x)$. This gives $z > x+(1-p)h > x_0$ and

$$G(z) = F(z)-\beta z < F(x)+\alpha(z-x)-\beta z = G(x)-(\beta-\alpha)(z-x)$$

$$< G(x)-(\beta-\alpha)(1-p)h < G(x_0),$$

which contradicts the definition of x_0. Consequently we may conclude that inequality (3.2.1) is valid.

But if (3.2.1) holds everywhere in the interval (c, d) then certainly $\bar{F}^*(t) \leqslant \beta$ for all points $t \in (c, d)$; but there are points from $\hat{X}^n_{p\alpha\beta}$ in that interval and at those points we must have $\bar{F}^*(t) > \beta$. As this is impossible it follows that each set $\hat{X}^n_{p\alpha\beta}$ is nowhere dense as claimed and the theorem now follows.

We have been unable to determine whether Theorem 3.2 is sharp. Specifically, if $E = \{E_x : x \in \mathbb{R}\}$ is a system of paths for which each E_x is strongly porous on both sides at x, must there be a continuous function F for which the set $\{x : \bar{F}'_E(x) < \bar{F}^*(x)\}$ is second category? An affirmative answer would show that strong porosity is the correct concept with which to capture the nature of the W. H. Young relation.

4. *The Sierpiński–Young relations*

We turn now to the problem of obtaining generalized versions of Theorem II. In our setting this involves finding conditions that must be placed on a pair of systems $E = \{E_x : x \in \mathbb{R}\}$ and $E^* = \{E^*_x : x \in \mathbb{R}\}$ so that some conclusion about the inequalities $\underline{F}'_E(x) \leqslant \bar{F}'_{E^*}(x)$ and $\underline{F}'_{E^*}(x) \leqslant \bar{F}'_E(x)$ can be made.

The first result has appeared in [3, Theorem 4.2] and is an immediate generalization of Theorem II.

THEOREM 4.1. *Let $E = \{E_x : x \in \mathbb{R}\}$ and $E^* = \{E^*_x : x \in \mathbb{R}\}$ be systems of paths both of which satisfy the intersection condition. Then for an arbitrary function F the set of points x at which either $\bar{F}'_E(x) < \underline{F}'_{E^*}(x)$ or $\underline{F}'_E(x) > \bar{F}'_{E^*}(x)$ is denumerable.*

Proof. See [3].

While this theorem can be applied to a number of generalized derivatives it does not apply to the approximate Dini derivatives. Indeed as shown in [9] that theorem would be false even if F were taken as continuous. One could impose more severe restrictions on F as for example the assumption that it is Lipshitz; but for Lipshitz functions the approximate Dini derivatives are just the ordinary Dini derivatives anyway. Instead we weaken the conclusion to apply only to the set where all derivates are finite. Our next theorem generalizes a theorem of Zajíček [9] with a rather more direct proof.

THEOREM 4.2. *Let $E = \{E_x : x \in \mathbb{R}\}$ and $E^* = \{E^*_x : x \in \mathbb{R}\}$ be systems of paths both of which satisfy the one-sided sharp external intersection condition. Then the set of points x at which a function F has all four of $\underline{F}'_E(x)$, $\bar{F}'_E(x)$, $\underline{F}'_{E^*}(x)$ and $\bar{F}'_{E^*}(x)$ finite but either $\underline{F}'_E(x) > \bar{F}'_{E^*}(x)$ or $\bar{F}'_E(x) < \underline{F}'_{E^*}(x)$ is denumerable.*

Proof. For natural numbers m and for positive rationals α, β and γ define the set $X_{\alpha\beta\gamma m}$ as the collection of all points x at which

$$-m < \underline{F}'_E(x) \leqslant \bar{F}'_E(x) < \alpha < \beta < \gamma < \underline{F}'_{E^*}(x) \leqslant \bar{F}'_{E^*}(x) < m .$$

We shall show that each such set is denumerable. Similar arguments show that the same set with α, β, and γ negative rationals is also denumerable. It follows then, upon formation of an appropriate union, that the sets of points x at which all four derivates are finite, and yet $\bar{F}'_E(x) < \underline{F}'_{E^*}(x)$, is denumerable. Since E and E^* are interchangeable the theorem is proved.

To this end let $\varepsilon_1 = (\beta-\alpha)/(m+\alpha)$ and choose a positive function δ_1 such that the intersection condition of (2.3.2) is met for E and ε_1, and choose a positive function δ_2 such that for every $x \in X_{\alpha\beta\gamma m}$, $y \in E_x$ and $0 < |y-x| < \delta_2(x)$ one has

$$-m < \frac{F(y)-F(x)}{y-x} < \alpha .$$

Similarly let $\varepsilon_2 = (\gamma-\beta)/m$ and choose positive functions δ_3 and δ_4 such that δ_3 gives the intersection condition of (2.3.2) for the system E^* and the positive number ε_2, and such that for every $x \in X_{\alpha\beta\gamma m}$, $y \in E_x^*$ and $0 < |y-x| < \delta_4(x)$ one has

$$\gamma < \frac{F(y)-F(x)}{y-x} < m .$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4\}$ and let $\{X^n_{\alpha\beta\gamma m}\}$ be a δ-composition of the set $X_{\alpha\beta\gamma m}$. In order to show that each $X_{\alpha\beta\gamma m}$ is denumerable we have only to show that each member of the decomposition is finite. In fact each such set can contain at most a single element.

Suppose contrary to this that there are two points x and y, with $x < y$, in some set $X^n_{\alpha\beta\gamma m}$. By the nature of the decomposition there must be a point z in one of the two sets

$$E_x \cap E_y \cap [y, y+\varepsilon_1(y-x)) \qquad \text{and} \qquad E_x \cap E_y \cap (x-\varepsilon_1(y-x), x] .$$

There are a number of cases to consider: let us just take the situation with $z \in E_x \cap E_y$ and $y < z < y+\varepsilon_1(y-x)$ as the other cases are similar. We have then $F(z)-F(x) < \alpha(z-x)$ and $F(z)-F(y) > -m(z-y)$. Hence

$$F(y)-F(x) = F(y)-F(z)+F(z)-F(x) < m(z-y)+\alpha(z-x)$$

$$< \alpha(1+\varepsilon_1)(y-x)+m\varepsilon_1(y-x) = [\alpha(1+\varepsilon_1)+m\varepsilon_1](y-x) = \beta(y-x) .$$

In fact all cases will yield $F(y)-F(x) < \beta(y-x)$.

Arguing similarly for the system E^*, we see that there must be a point z' in one of the two sets

$$E_x^* \cap E_y^* \cap [y, y+\varepsilon_2(y-x)) \qquad \text{and} \qquad E_x^* \cap E_y^* \cap (x-\varepsilon_2(y-x), x] .$$

In the same way one obtains the inequality $F(y)-F(x) > \beta(y-x)$.

As the two inequalities conflict we must conclude that such pairs of points cannot exist, so that each set $X^n_{\alpha\beta\gamma m}$ is either empty or contains a single point. As remarked above the theorem now follows.

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