Chapter 6

MORE ON CONTINUOUS FUNCTIONS AND SETS

This chapter can be considered enrichment material containing also several more advanced topics and may be skipped in its entirety. You can proceed directly to the study of derivatives and integrals with no loss in the continuity of the material.

6.1 Introduction

In this chapter we go much more deeply into the analysis of continuous functions. For this we need some new set theoretic ideas and methods.

6.2 Dense Sets

[This section reviews material from Section 1.9.]

Consider the set $\mathbb{Q}$ of rational numbers and let $(a, b)$ be an open interval in $\mathbb{R}$. How do we show that there is a member of $\mathbb{Q}$ in the interval $(a, b)$; that is, that $(a, b) \cap \mathbb{Q} \neq \emptyset$?
Suppose first that \(0 < a\). Since \(b - a > 0\), the archimedean property (Theorem 1.11) implies that there is a positive integer \(q\) such that
\[
q(b - a) > 1.
\]
Thus
\[
qb > 1 + qa.
\]
The archimedean property also implies that the set of integers
\[
\{m \in \mathbb{N} : m > qa\}
\]
is nonempty. Thus, according to the well-ordering principle, there is a smallest integer \(p\) in this set and for this \(p\), it is true that \(p - 1 \leq qa < p\). It follows that
\[
qa < p \leq 1 + qa < qb,
\]
which implies \(a < \frac{p}{q} < b\). We have shown that, under the assumption \(a > 0\), there exists a rational number \(r = \frac{p}{q}\) in the interval \((a, b)\).

The same is true under the assumption \(a < 0\). To see this observe first that if \(a < 0 < b\), we can take \(r = 0\). If \(a < b < 0\), then \(0 < -b < -a\), so the argument of the previous paragraph shows that there exists \(r \in \mathbb{Q}\) such that \(-b < r < -a\). In this case \(a < -r < b\).

The preceding discussion proves that every open interval contains a rational number. We often express this fact by saying that the set of rational numbers is a dense set.

**Definition 6.1:** A set of real numbers \(A\) is said to be dense (in \(\mathbb{R}\)) if for each open interval \((a, b)\) the set \(A \cap (a, b)\) is nonempty.

It is important to have a more general concept, that of a set \(A\) being dense in a set \(B\).

**Definition 6.2:** Let \(A\) and \(B\) be subsets of \(\mathbb{R}\). If every open interval that intersects \(B\) also intersects \(A\), we say that \(A\) is dense in \(B\).
Thus Definition 6.1 states the special case of Definition 6.2 that occurs when $B = \mathbb{R}$. We should note that some authors require that $A \subset B$ in their version of Definition 6.2. We find it more convenient not to impose this restriction. Thus, for example, in our language $\mathbb{Q}$ is dense in $\mathbb{R} \setminus \mathbb{Q}$.

It is easy to verify that $A$ is dense in $B$ if and only if $\overline{A} \supset B$ (Exercise 6.2.1).

**Exercises**

6.2.1 Verify that $A$ is dense in $B$ if and only if $\overline{A} \supset B$.

6.2.2 Prove that every set $A$ is dense in its closure $\overline{A}$.

6.2.3 Prove that if $A$ is dense in $B$ and $C \subset B$, then $A$ is dense in $C$.

6.2.4 Prove that if $A \subset B$ and $A$ is dense in $B$, then $\overline{A} = \overline{B}$. Is the statement correct without the assumption that $A \subset B$?

6.2.5 Is $\mathbb{R} \setminus \mathbb{Q}$ dense in $\mathbb{Q}$?

6.2.6 The following are several pairs $(A, B)$ of sets. In each case determine whether $A$ is dense in $B$.

(a) $A = \mathbb{N}, B = \mathbb{N}$

(b) $A = \mathbb{N}, B = \mathbb{Z}$

(c) $A = \mathbb{N}, B = \mathbb{Q}$

(d) $A = \{x : x = \frac{m}{2n}, m \in \mathbb{Z}, n \in \mathbb{N}\}, B = \mathbb{Q}$

6.2.7 Let $A$ and $B$ be subsets of $\mathbb{R}$. Prove that $A$ is dense in $B$ if and only if for every $b \in B$ there exists a sequence $\{a_n\}$ of points from $A$ such that $\lim_{n \to \infty} a_n = b$.

6.2.8 Let $B$ be the set of all irrational numbers. Prove that the set

$$A = \{q + \sqrt{2} : q \in \mathbb{Q}\}$$

is a countable subset of $B$ that is dense in $B$.

6.2.9 Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing continuous function. Does $f$ map dense sets to dense sets; that is, is it true that

$$f(E) = \{f(x) : x \in E\}$$
is dense if $E$ is dense?

See Note 147

6.2.10 Prove that every set $B \subset \mathbb{R}$ contains a countable set $A$ that is dense in $B$.

6.3 Nowhere Dense Sets

We might view a set $A$ that is dense in $\mathbb{R}$ as being somehow large: Inside every interval, no matter how small, we find points of $A$. There is an opposite extreme to this situation: A set is said to be nowhere dense, and hence is in some sense small, if it is not dense in any interval at all. The precise definition of this important concept of smallness follows.

**Definition 6.3:** The set $A \subset \mathbb{R}$ is said to be nowhere dense in $\mathbb{R}$ provided every open interval $I$ contains an open subinterval $J$ such that $A \cap J = \emptyset$.

We can state this another way: $A$ is nowhere dense provided $\overline{A}$ contains no open intervals. (See Exercise 6.3.4.)

**Example 6.4:** It is easy to construct examples of nowhere dense sets.

1. Any finite set
2. $\mathbb{N}$
3. $\{1/n : n \in \mathbb{N}\}$

Each of these sets is nowhere dense, as you can verify.

Each of the sets in Example 6.4 is countable and hence also small in the sense of cardinality. It is hard to imagine an uncountable set that is nowhere dense but, as we shall see in Section 6.5, such sets do exist.

We establish a simple result showing that any finite union of nowhere dense sets is again nowhere dense. It is not true that a countable union of nowhere dense sets is again nowhere dense. Indeed countable unions of nowhere dense sets will be important in our subsequent study.

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**Theorem 6.5:** Let \( A_1, A_2, \ldots, A_n \) be nowhere dense in \( \mathbb{R} \). Then \( A_1 \cup \cdots \cup A_n \) is also nowhere dense in \( \mathbb{R} \).

**Proof.** Let \( I \) be any open interval in \( \mathbb{R} \). We seek an open interval \( J \subset I \) such that \( J \cap A_i = \emptyset \) for \( i = 1, 2, \ldots, n \).

Since \( A_1 \) is nowhere dense, there exists an open interval \( I_1 \subset I \) such that \( I_1 \cap A_1 = \emptyset \). Now \( A_2 \) is also nowhere dense in \( \mathbb{R} \), so there exists an open interval \( I_2 \subset I_1 \) such that \( A_2 \cap I_2 = \emptyset \). Proceeding in this way we obtain open intervals

\[
I_1 \supset I_2 \supset I_3 \supset \cdots \supset I_n
\]

such that for \( i = 1, \ldots, n \), \( A_i \cap I_i = \emptyset \). It follows from the fact that \( I_n \subset I_i \) for \( i = 1, \ldots, n \) that \( A_i \cap I_n = \emptyset \) for \( i = 1, \ldots, n \). Thus

\[
\left( \bigcup_{i=1}^{n} A_i \right) \cap I_n = \bigcup_{i=1}^{n} (A_i \cap I_n) = \bigcup_{i=1}^{n} \emptyset = \emptyset,
\]

as was to be proved. \( \blacksquare \)

**Exercises**

6.3.1 Give an example of a sequence of nowhere dense sets whose union is not nowhere dense.

6.3.2 Which of the following statements are true?

(a) Every subset of a nowhere dense set is nowhere dense.
(b) If \( A \) is nowhere dense, then so too is \( A + c = \{ t + c : t \in A \} \) for every number \( c \).
(c) If \( A \) is nowhere dense, then so too is \( cA = \{ ct : t \in A \} \) for every positive number \( c \).
(d) If \( A \) is nowhere dense, then so too is \( A' \), the set of derived points of \( A \).
(e) A nowhere dense set can have no interior points.
(f) A set that has no interior points must be nowhere dense.
(g) Every point in a nowhere dense set must be isolated.
(h) If every point in a set is isolated, then that set must be nowhere dense.

See Note 149

6.3.3 If $A$ is nowhere dense, what can you say about $\mathbb{R} \setminus A$? If $A$ is dense, what can you say about $\mathbb{R} \setminus A$?

6.3.4 Prove that a set $A \subset \mathbb{R}$ is nowhere dense if and only if $\overline{A}$ contains no intervals; equivalently, the interior of $\overline{A}$ is empty.

6.3.5 What should the statement “$A$ is nowhere dense in the interval $I$” mean? Give an example of a set that is nowhere dense in $[0, 1]$ but is not nowhere dense in $\mathbb{R}$.

6.3.6 Let $A$ and $B$ be subsets of $\mathbb{R}$. What should the statement “$A$ is nowhere dense in the $B$” mean? Is $\mathbb{N}$ nowhere dense in $[0, 10]$? Is $\mathbb{N}$ nowhere dense in $\mathbb{Z}$? Is $\{4\}$ nowhere dense in $\mathbb{N}$?

6.3.7 Prove that the complement of a dense open subset of $\mathbb{R}$ is nowhere dense in $\mathbb{R}$.

6.3.8 Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing continuous function. Show that $f$ maps nowhere dense sets to nowhere dense sets; that is,

$$f(E) = \{f(x) : x \in E\}$$

is nowhere dense if $E$ is nowhere dense.

6.4 The Baire Category Theorem

$\Rightarrow$ Advanced section. May be omitted.

In this section we shall establish the Baire category theorem, which gives a sense in which nowhere dense sets can be viewed as “small”: A union of a sequence of nowhere dense sets cannot fill up an interval. If we interpret Cantor’s theorem (Theorem 2.4) as asserting that a union of a sequence of finite sets cannot fill up an interval, then we see the Baire category theorem as a far-reaching generalization.

We motivate this important theorem by way of a game idea that is due to Stefan Banach (1892–1945) and Stanislaw Mazur (1905–1981). Although the origins of the theorem are due to René Baire, after whom the theorem is named, the game approach helps us see why the Baire category theorem might be true. This Banach-Mazur game is just one of many mathematical games that are used throughout mathematics to develop interesting concepts.
6.4.1 A Two-Player Game

We introduce the Baire category theorem via a game between two players (A) and (B).

Player (A) is given a subset $A$ of $\mathbb{R}$, and player (B) is given the complementary set $B = \mathbb{R} \setminus A$. Player (A) first selects a closed interval $I_1 \subset \mathbb{R}$; then player (B) chooses a closed interval $I_2 \subset I_1$. The players alternate moves, a move consisting of selecting a closed interval inside the previously chosen interval.

The play of the game thus determines a descending sequence of closed intervals

$$I_1 \supset I_2 \supset I_3 \supset \cdots \supset I_n \supset \cdots$$

where player (A) chooses those with odd index and player (B) those with even index. If

$$A \cap \bigcap_{n=1}^{\infty} I_n \neq \emptyset,$$

then player (A) wins; otherwise player (B) wins.

The goal of player (A) is evidently to make sure that the intersection contains a point of $A$; the goal of player (B) is to ensure that the intersection is empty or contains only points of $B$. We expect that player (A) should win if his set $A$ is large while player (B) should win if his set is large. It is not, however, immediately clear what “large” might mean for this game.

Example 6.6: If the set $A$ given to player (A) contains an open interval $J$, then (A) should choose any interval $I_1 \subset J$. No matter how the game continues, player (A) wins. Another way to say this: If the set given to player (B) is not dense, he loses.

Example 6.7: For a more interesting example, let player (A) be dealt the “large” set of all irrational numbers, so that player (B) is dealt the rationals. (Both players have been dealt dense sets now.) Let $A$ consist of the irrational numbers. Player (A) can win by following the strategy we now describe. Let $q_1, q_2, q_3, \ldots$ be a listing of all of the rational numbers; that is,

$$\mathbb{Q} = \{q_1, q_2, q_3, \ldots\}.$$
Player (A) chooses the first interval $I_1$ as any closed interval such that $q_1 \notin I_1$. Inductively, suppose $I_1, I_2, \ldots, I_{2n}$ have been chosen according to the rules of the game so that it is now time for player (A) to choose $I_{2n+1}$. The set $\{q_1, q_2, \ldots, q_n\}$ is finite, so there exists a closed interval $I_{2n+1} \subset I_{2n}$ such that

$$I_{2n+1} \cap \{q_1, q_2, \ldots, q_n\}$$

is empty. Player (A) chooses such an interval.

Since for each $n \in \mathbb{N}$, $q_n \notin I_{2n+1}$, the set $\bigcap_{n=1}^{\infty} I_n$ contains no rational numbers, but, as a descending sequence of closed intervals, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. Thus $A \cap \bigcap_{n=1}^{\infty} I_n \neq \emptyset$, and (A) wins.

In these two examples, using informal language, we can say that player (A) has a strategy to win: No matter how player (B) proceeds, player (A) can “answer” each move and win the game.

In both examples player (A) had a clear advantage: The set $A$ was larger than the set $B$. But in what sense is it larger? It is not the fact that $A$ is uncountable while $B$ is countable that matters here. It is something else: The fact that given an interval $I_{2n}$, player (A) can choose $I_{2n+1}$ inside $I_{2n}$ in such a way that $I_{2n+1}$ misses the set $\{q_1, q_2, \ldots, q_n\}$.

Let us try to see in the second example a general strategy that should work for player (A) in some cases. The set $B$ was the union of the singleton sets $\{q_n\}$. Suppose instead that $B$ is the union of a sequence of “small” sets $Q_n$. Then the same “strategy” will prevail if given any interval $J$ and given any $n \in \mathbb{N}$, there exists an interval $I \subset J$ such that

$$I \cap (Q_1 \cup Q_2 \cup \cdots \cup Q_n) = \emptyset.$$

The set $\bigcap_{n=1}^{\infty} I_n$ will be nonempty, and will miss the set $\bigcup_{n=1}^{\infty} Q_n$. Thus, if $B = \bigcup_{n=1}^{\infty} Q_n$, player (A) has a winning strategy. It is in this sense that the set $B$ is “small.” The set $A$ is “large” because the set $B$ is “small”. If we look carefully at the requirement on the sets $Q_k$, we see it is just that each of these sets is nowhere dense in $\mathbb{R}$.

Thus the key to player (A) winning rests on the concept of a nowhere dense set. But note that it rests on the set $B$ being the union of a sequence of nowhere dense sets.
We can formulate our result from our discussion of the game in several ways:

1. \( \mathbb{R} \) cannot be expressed as a countable union of nowhere dense sets.

2. The complement of a countable union of nowhere dense sets is dense.

The second of these provides a sense in which countable unions of nowhere dense sets are “small:” No matter which countable collection of nowhere dense sets we choose, their union leaves a dense set uncovered.

To formulate the Baire category theorem we need some definitions. This is the original language of Baire and it has survived; he simply places sets in two types or categories. Into the first category he places the sets that are to be considered small and into the second category he puts the remaining (not small) sets.

**Definition 6.8:** Let \( A \) be a set of real numbers.

1. \( A \) is said to be of the **first category** if it can be expressed as a countable union of nowhere dense sets.

2. \( A \) is said to be of the **second category** if it is not of the first category.

3. \( A \) is said to be **residual** in \( \mathbb{R} \) if the complement \( \mathbb{R} \setminus A \) is of the first category.

The following properties of first category sets and their complements, the residual sets, are easily proved and left as exercises.

**Lemma 6.9:** A union of any sequence of first category sets is again a first category set.

**Lemma 6.10:** An intersection of any sequence of residual sets is again a residual set.
Theorem 6.11 (Baire Category Theorem) Every residual subset of $\mathbb{R}$ is dense in $\mathbb{R}$.

Proof. The discussion in Section 6.4.1 constitutes a proof. Suppose that player (A) is dealt a set $A = X \cap [a, b]$ where $X$ is residual. Then there is a sequence of nowhere dense sets $\{Q_n\}$ so that

$$X = \mathbb{R} \setminus \bigcup_{n=1}^{\infty} Q_n.$$  

Then player (A) wins by choosing any interval $I_1 \subset [a, b]$ that avoids $Q_1$ and continues following the strategy of Section 6.4.1. In particular, $X$ must contain a point of the interval $[a, b]$, and hence a point of any interval.

Theorem 6.11 provides a sense of largeness of sets that is not shared by dense sets in general. The intersection of two dense sets might be empty, but the intersection of two, or even countably many, residual sets must still be dense.

Exercises

6.4.1 Show that the union of any sequence of first category sets is again a first category set.

6.4.2 Show that the intersection of any sequence of residual sets is again a residual set.

6.4.3 Rewrite the proof of Theorem 6.11 without using the games language.

6.4.4 Give an example of two dense sets whose intersection is not dense. Does this contradict Theorem 6.11?

6.4.5 Suppose that $\bigcup_{n=1}^{\infty} A_n$ contains some interval $(c, d)$. Show that there is a set, say $A_{n_0}$, and a subinterval $(c', d') \subset (c, d)$ so that $A_{n_0}$ is dense in $(c', d').$
6.4.3 Uniform Boundedness

There are many applications of the Baire category Theorem in analysis. For now, we present just one application, dealing with the concept of uniform boundedness. Suppose we have a collection $F$ of functions defined on $\mathbb{R}$ with the property that for each $x \in \mathbb{R}$, $\{|f(x)| : f \in F\}$ is bounded. This means that for each $x \in \mathbb{R}$, there exists a number $M_x \geq 0$ such that $|f(x)| \leq M_x$ for all $f \in F$. We can describe this situation by saying that $F$ is pointwise bounded. Does this imply that the collection is uniformly bounded; that is, that there is a single number $M$ so that $|f(x)| \leq M$ for all $f \in F$ and every $x \in \mathbb{R}$?

Example 6.12: Let $q_1, q_2, q_3, \ldots$ be an enumeration of $\mathbb{Q}$. For each $n \in \mathbb{N}$ we define a function $f_n$ by $f_n(q_k) = k$ if $n \leq k$, $f_n(x) = 0$ for all other values $x$. Let $F = \{f_n : n \in \mathbb{N}\}$. Then if $x \in \mathbb{R} \setminus \mathbb{Q}$, $f(x) = 0$ for all $f \in F$, and if $x = q_k$, $|f(x)| \leq k$ for all $f \in F$. Thus, for each $x \in \mathbb{R}$, the set $\{|f(x)| : f \in F\}$ is bounded. The bounds can be taken to be 0 if $x \in \mathbb{R} \setminus \mathbb{Q}$ ($M_x = 0$ if $x \in \mathbb{R} \setminus \mathbb{Q}$) and we can take $M_{q_k} = k$. But since $\mathbb{Q}$ is dense in $\mathbb{R}$, none of the functions $f_n$ is bounded on any interval. (Verify this.) Thus a collection of functions may be pointwise bounded but not uniformly bounded on any interval.

The functions $f_n$ in Example 6.12 are everywhere discontinuous. Our next theorem shows that if we had taken a collection $F$ of continuous functions, then not only would each $f \in F$ be bounded on closed intervals (as Theorem 5.49 guarantees), but there would be an interval $I$ on which the entire collection is uniformly bounded; that is, there exists a constant $M$ such that $|f(x)| \leq M$ for all $f \in F$ and each $x \in I$.

Theorem 6.13: Let $F$ be a collection of continuous functions on $\mathbb{R}$ such that for each $x \in \mathbb{R}$ there exists a constant $M_x > 0$ such that $|f(x)| \leq M_x$ for each $f \in F$. Then there exists an open interval $I$ and a constant $M > 0$ such that $|f(x)| \leq M$ for each $f \in F$ and each $x \in I$.

Proof. For each $n \in \mathbb{N}$, let $A_n = \{x : |f(x)| \leq n \text{ for all } f \in F\}$. By hypothesis, $\mathbb{R} = \bigcup_{n=1}^{\infty} A_n$. Also, by hypothesis, each $f \in F$ is continuous and so it is easy to check that each of the sets

$$\{x : |f(x)| \leq n\}$$
must be closed (e.g., Exercise 5.4.31). Thus

\[ A_n = \bigcap_{f \in \mathcal{F}} \{ x : |f(x)| \leq n \} \]

is an intersection of closed sets and is therefore itself closed. This expresses the real line \( \mathbb{R} \) as a union of the sequence of closed sets \( \{ A_n \} \).

It now follows from the Baire category theorem that at least one of the sets, say \( A_{n_0} \), must be dense in some open interval \( I \). Since \( A_{n_0} \) is closed and dense in the interval \( I \), \( A_{n_0} \) must contain \( I \). This means that \( |f(x)| \leq n_0 \) for each \( f \in \mathcal{F} \) and all \( x \in I \). ■

**Exercises**

6.4.6 Let \( \{ f_n \} \) be a sequence of continuous functions on an interval \([a, b]\) such that

\[ \lim_{n \to \infty} f_n(x) = f(x) \]

exists at every point \( x \in [a, b] \). Show that \( f \) need not be continuous nor even bounded, but that \( f \) must be bounded on some subinterval of \([a, b]\).

6.4.7 Let \( \{ f_n \} \) be a sequence of continuous functions on \([0, 1]\) and suppose that

\[ \lim_{n \to \infty} f_n(x) = 0 \]

for all \( 0 \leq x \leq 1 \). Show that there must be an interval \([c, d] \subset [0, 1]\) so that, for all sufficiently large \( n \), \( |f_n(x)| \leq 1 \) for all \( x \in [c, d] \).

See Note 155

6.4.8 Give an example of a sequence of functions on \([0, 1]\) with the property that

\[ \lim_{n \to \infty} f_n(x) = 0 \]

for all \( 0 \leq x \leq 1 \) and yet for every interval \([c, d] \subset [0, 1]\) and every \( N \) there is some \( x \in [c, d] \) and \( n > N \) with \( f_n(x) > 1 \).
6.5 Cantor Sets

We say that a set is perfect if it is a nonempty closed set with no isolated points. The only examples that might come to mind are sets that are finite unions of intervals. It might be difficult to imagine a perfect subset of $\mathbb{R}$ that is also nowhere dense. In this section we obtain such a set, the very important classical Cantor set. We also discuss some of its variants. Such sets have historical significance and are of importance in a number of areas of mathematical analysis.

6.5.1 Construction of the Cantor Ternary Set

We begin with the closed interval $[0,1]$. From this interval we shall remove a dense open set $G$. The remaining set $K = [0,1] \setminus G$ will then be closed and nowhere dense in $[0,1]$. We construct $G$ in such a way that $K$ has no isolated points and is nonempty. Thus $K$ will be a nonempty, nowhere dense perfect subset of $[0,1]$.

It is easiest to understand the set $G$ if we construct it in stages. Let $G_1 = \left(\frac{1}{3}, \frac{2}{3}\right)$, and let $K_1 = [0,1] \setminus G_1$. Thus $K_1 = [0, \frac{1}{3}] \cup \left[\frac{2}{3}, 1\right]$ is what remains when the middle third of the interval $[0,1]$ is removed. This is the first stage of our construction.

We repeat this construction on each of the two component intervals of $K_1$. Let $G_2 = \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right)$ and let $K_2 = [0,1] \setminus (G_1 \cup G_2)$. Thus

$$K_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right].$$

This completes the second stage.

We continue inductively, obtaining two sequences of sets, $\{K_n\}$ and $\{G_n\}$ with the following properties: For each $n \in \mathbb{N}$

1. $G_n$ is a union of $2^{n-1}$ pairwise disjoint open intervals.
2. $K_n$ is a union of $2^n$ pairwise disjoint closed intervals.
3. \( K_n = [0, 1] \setminus (G_1 \cup G_2 \cup \cdots \cup G_n) \).

4. Each component of \( G_{n+1} \) is the “middle third” of some component of \( K_n \).

5. The length of each component of \( K_n \) is \( \frac{1}{3^n} \).

Figure 6.1 shows \( K_1, K_2, \) and \( K_3 \).

Now let

\[
G = \bigcup_{n=1}^{\infty} G_n
\]

and let

\[
K = [0, 1] \setminus G = \bigcap_{n=1}^{\infty} K_n.
\]

Then \( G \) is open and the set \( K \) (our Cantor set) is closed.

To see that \( K \) is nowhere dense, it is enough, since \( K \) is closed, to show that \( K \) contains no open intervals (Exercise 6.3.4). Let \( J \) be an open interval in \([0,1]\) and let \( \lambda \) be its length. Choose \( n \in \mathbb{N} \) such that \( \frac{1}{3^n} < \lambda \). By property 5, each component of \( K_n \) has length \( \frac{1}{3^n} < \lambda \), and by property 2 the components
of $K_n$ are pairwise disjoint. Thus $K_n$ cannot contain $J$, so neither can $K = \bigcap_1^\infty K_n$. We have shown that the closed set $K$ contains no intervals and is therefore nowhere dense.

It remains to show that $K$ has no isolated points. Let $x_0 \in K$. We show that $x_0$ is a limit point of $K$. To do this we show that for every $\varepsilon > 0$ there exists $x_1 \in K$ such that $0 < |x_1 - x_0| < \varepsilon$. Choose $n$ such that $1/3^n < \varepsilon$. There is a component $L$ of $K_n$ that contains $x_0$. This component is a closed interval of length $1/3^n < \varepsilon$. The set $K_{n+1} \cap L$ has two components $L_0$ and $L_1$, each of which contains points of $K$. The point $x_0$ is in one of the components, say $L_0$. Let $x_1$ be any point of $K \cap L_1$. Then $0 < |x_0 - x_1| < \varepsilon$. This verifies that $x_0$ is a limit point of $K$. Thus $K$ has no isolated points.

The set $K$ is called the Cantor set. Because of its construction, it is often called the Cantor middle third set. In a moment we shall present a purely arithmetic description of the Cantor set that suggests another common name for $K$, the “Cantor ternary set”. But first, we mention a few properties of $K$ and of its complement $G$ that may help you visualize these sets.

First note that $G$ is an open dense set in $[0,1]$. Write $G = \bigcup_{k=1}^\infty (a_k, b_k)$. (The component intervals $(a_k, b_k)$ of $G$ can be called the intervals complementary to $K$ in $(0,1)$.) Each is a middle third of a component interval of some $K_n$.) Observe that no two of these component intervals can have a common endpoint. If, for example, $b_m = a_n$, then this point would be an isolated point of $K$, and $K$ has no isolated points.

Next observe that for each $k \in \mathbb{N}$, the points $a_k$ and $b_k$ are points of $K$. But there are other points of $K$ as well. In fact, we shall see presently that $K$ is uncountable. These other points are all limit points of the endpoints of the complementary intervals. The set of endpoints is countable, but the closure of this set is uncountable as we shall see. Thus, in the sense of cardinality, “most” points of the Cantor set are not endpoints of intervals complementary to $K$.

Each component interval of the set $G_n$ has length $1/3^n$; thus the sum of the lengths of these component intervals is

$$\frac{2^{n-1}}{3^n} = \frac{1}{2} \left(\frac{2}{3}\right)^n.$$
It follows that the lengths of all component intervals of $G$ forms a geometric series with sum

$$\sum_{n=1}^{\infty} \frac{1}{2} \left( \frac{2}{3} \right)^n = 1.$$ 

(This also gives us a clue as to why $K$ cannot contain an interval: After removing from the unit interval a sequence of pairwise disjoint intervals with length-sum one, no room exists for any intervals in the set $K$ that remains.)

**Exercises**

6.5.1 Let $E$ be the set of endpoints of intervals complementary to the Cantor set $K$. Prove that $\overline{E} = K$.

6.5.2 Let $G$ be a dense open subset of $\mathbb{R}$ and let $\{(a_k, b_k)\}$ be its set of component intervals. Prove that $H = \mathbb{R} \setminus G$ is perfect if and only if no two of these intervals have common endpoints.

6.5.3 Let $K$ be the Cantor set and let $\{(a_k, b_k)\}$ be the sequence of intervals complementary to $K$ in $[0, 1]$. For each $k \in \mathbb{N}$, let $c_k = (a_k + b_k)/2$ (the midpoint of the interval $(a_k, b_k)$) and let $N = \{c_k : k \in \mathbb{N}\}$. Prove each of the following:

   (a) Every point of $N$ is isolated.
   (b) If $c_i \neq c_j$, there exists $k \in \mathbb{N}$ such that $c_k$ is between $c_i$ and $c_j$ (i.e., no point in $N$ has an immediate “neighbor” in $N$).
   (c) Show that there is an order-preserving mapping $\phi : \mathbb{Q} \cap (0, 1) \to N$ [i.e., if $x < y \in \mathbb{Q} \cap (0, 1)$, then $\phi(x) < \phi(y) \in N$]. This may seem surprising since $\mathbb{Q} \cap (0, 1)$ has no isolated points while $N$ has only isolated points.

6.5.4 It is common now to say that a set $E$ of real numbers is a Cantor set if it is nonempty, bounded, perfect, and nowhere dense. Show that the union of a finite number of Cantor sets is also a Cantor set.

6.5.5 Show that every Cantor set is uncountable.

6.5.6 Let $A$ and $B$ be subsets of $\mathbb{R}$. A function $h$ that maps $A$ onto $B$, is one-to-one, and with both $h$ and $h^{-1}$ continuous is called a homeomorphism between $A$ and $B$. The sets $A$ and $B$ are said to be homeomorphic. Prove that a set $C$ is a Cantor set if and only if it is homeomorphic to the Cantor ternary set $K$. 
6.5.2 An Arithmetic Construction of $K$

We turn now to a purely arithmetical construction for the Cantor set. You will need some familiarity with ternary (base 3) arithmetic here.

Each $x \in [0,1]$ can be expressed in base 3 as

$$x = .a_1a_2a_3 \ldots,$$

where $a_i = 0, 1$ or $2, i = 1,2,3, \ldots$. Certain points have two representations, one ending with a string of zeros, the other in a string of twos. For example, $.100 \cdots = .0222 \cdots$ both represent the number $1/3$ (base ten). Now, if $x \in (1/3, 2/3)$, $a_1 = 1$, thus each $x \in G_1$ must have ‘1’ in the first position of its ternary expansion. Similarly, if

$$x \in G_2 = \left( \frac{1}{9}, \frac{2}{9} \right) \cup \left( \frac{7}{9}, \frac{8}{9} \right),$$

it must have a 1 in the second position of its ternary expansion (i.e., $a_2 = 1$). In general, each point in $G_n$ must have $a_n = 1$. It follows that every point of $G = \bigcup_1^\infty G_n$ must have a 1 someplace in its ternary expansion.

Now endpoints of intervals complementary to $K$ have two representations, one of which involves no 1’s. The remaining points of $K$ never fall in the middle third of a component of one of the sets $K_n$, and so have ternary expansions of the form

$$x = .a_1a_2 \ldots \ a_i = 0 \ or \ 2.$$ 

We can therefore describe $K$ arithmetically as the set

$$\{x = .a_1a_2a_3 \ldots \ (\text{base three}) : a_i = 0 \ or \ 2 \ for \ each \ i \in \mathbb{N}\}.$$

As an immediate result, we see that $K$ is uncountable. In fact, $K$ can be put into 1-1 correspondence with $[0,1]$: For each

$$x = .a_1a_2a_3 \ldots \ (\text{base 3}), \ a_i = 0,2,$$
in the set $K$, let there correspond the number
\[ y = .b_1 b_2 b_3 \ldots \text{ (base 2)}, \quad b_i = a_i / 2. \]
This provides a 1-1 correspondence between $K$ (minus endpoints of complementary intervals) and $[0, 1]$ (minus the countable set of numbers with two base 2 representations). By allowing these two countable sets to correspond to each other, we obtain a 1-1 correspondence between $K$ and $[0, 1]$.

**Note.** We end this section by mentioning that variations in the constructions of $K$ can lead to interesting situations. For example, by changing the construction slightly, we can remove intervals in such a way that
\[ G' = \bigcup_{k=1}^{\infty} (a'_k, b'_k) \]
with
\[ \sum_{k=1}^{\infty} (b'_k - a'_k) = 1/2 \]
(instead of 1), while still keeping $K' = [0, 1] \setminus G'$ nowhere dense and perfect. The resulting set $K'$ created problems for late nineteenth-century mathematicians trying to develop a theory of measure. The “measure” of $G'$ should be 1/2; the “measure” of $[0, 1]$ should be 1. Intuition requires that the measure of the nowhere dense set $K'$ should be $1 - \frac{1}{2} = \frac{1}{2}$. How can this be when $K'$ is so “small?”

**Exercises**

**6.5.7** Find a specific irrational number in the Cantor ternary set. 

**6.5.8** Show that the Cantor ternary set can be defined as
\[ K = \left\{ x \in [0, 1] : x = \sum_{n=1}^{\infty} \frac{i_n}{3^n} \text{ for } i_n = 0 \text{ or } 2 \right\}. \]
6.5.9 Let

\[ D = \left\{ x \in [0, 1] : x = \sum_{n=1}^{\infty} \frac{j_n}{3^n} \text{ for } j_n = 0 \text{ or } 1 \right\}. \]

Show that \( D + D = \{ x + y : x, y \in D \} = [0, 1] \). From this deduce, for the Cantor ternary set \( K \), that \( K + K = [0, 2] \).

6.5.10 A careless student makes the following argument. Explain the error.

“If \( G = (a, b) \), then \( \overline{G} = [a, b] \). Similarly, if \( G = \bigcup_{i=1}^{\infty} (a_i, b_i) \) is an open set, then \( \overline{G} = \bigcup_{i=1}^{\infty} [a_i, b_i] \).

It follows that an open set \( G \) and its closure \( \overline{G} \) differ by at most a countable set.”

See Note 157

6.5.3 The Cantor Function

The Cantor set allows the construction of a rather bizarre function that is continuous and nondecreasing on the interval \([0, 1]\). It has the property that it is constant on every interval complementary to the Cantor set and yet manages to increase from \( f(0) = 0 \) to \( f(1) = 1 \) by doing all of its increasing on the Cantor set itself. It has sometimes been called “the devil’s staircase.”

Define the function \( f \) in the following way. On \((1/3, 2/3)\), let \( f = 1/2 \); on \((1/9, 2/9)\), let \( f = 1/4 \); on \((7/9, 8/9)\), let \( f = 3/4 \). Proceed inductively. On the \( 2^n - 1 \) open intervals appearing at the \( n \)th stage, define \( f \) to satisfy the following conditions:

1. \( f \) is constant on each of these intervals.

2. \( f \) takes the values

\[ \frac{1}{2^n}, \frac{3}{2^n}, \ldots, \frac{2^n-1}{2^n} \]

on these intervals.

3. If \( x \) and \( y \) are members of different \( n \)th-stage intervals with \( x < y \), then \( f(x) < f(y) \).
This description defines $f$ on $G = [0,1] \setminus K$. Extend $f$ to all of $[0,1]$ by defining $f(0) = 0$ and, for $0 < x \leq 1$,

$$f(x) = \sup \{ f(t) : t \in G, t < x \}.$$ 

In order to check that this defines the function that we want, we need to check each of the following.

1. $f(G)$ is dense in $[0,1]$.
2. $f$ is nondecreasing on $[0,1]$.
3. $f$ is continuous on $[0,1]$.
4. $f(K) = [0,1]$.

These have been left as exercises.

Figure 6.2 illustrates the construction. The function $f$ is called the Cantor function. Observe that $f$ “does all its rising” on the set $K$.

The Cantor function allows a negative answer to many questions that might be asked about functions and derivatives and, hence, has become a popular counterexample. For example, let us follow this kind of reasoning. If $f$ is a continuous function on $[0,1]$ and $f'(x) = 0$ for every $x \in (0,1)$ then $f$ is constant. (This is proved in most calculus courses by using the mean value theorem.) Now suppose that we know less, that $f'(x) = 0$ for every $x \in (0,1)$ excepting a “small” set $E$ of points at which we know nothing. If $E$ is finite it is still easy to show that $f$ must be constant. If $E$ is countable it is possible, but a bit more difficult, to show that it is still true that $f$ must be constant. The question then arises, just how small a set $E$ can appear here; that is, what would we have to know about a set $E$ so that we could say $f'(x) = 0$ for every $x \in (0,1) \setminus E$ implies that $f$ is constant?

The Cantor function is an example of a function constant on every interval complementary to the Cantor set $K$ (and so with a zero derivative at those points) and yet is not constant. The Cantor set, since it is nowhere dense, might be viewed as extremely small, but even so it is not insignificant for this problem.
Section 6.5. Cantor Sets

Figure 6.2. The third stage in the construction of the Cantor function.

Exercises

6.5.11 In the construction of the Cantor function complete the verification of details.

(a) Show that \( f(G) \) is dense in \([0, 1]\).
(b) Show that \( f \) is nondecreasing on \([0, 1]\).
(c) Infer from (a) and (b) that \( f \) is continuous on \([0, 1]\).
(d) Show that \( f(K) = [0, 1] \) and thus (again) conclude that \( K \) is uncountable.

6.5.12 Find a calculus textbook proof for the statement that a continuous function \( f \) on an interval \([a, b]\) that has a
zero derivative on \((a, b)\) must be constant. Improve the proof to allow a finite set of points on which \(f\) is not known to have a zero derivative.

### 6.6 Borel Sets

\footnote{Advanced section. May be omitted.}

In our study of continuous functions we have seen that the classes of open sets and closed sets play a significant role. But the class of sets that are of importance in analysis goes beyond merely the open and closed sets. E. Borel (1871–1956) recognized that for many operations of analysis we need to form countable intersections and countable unions of classes of sets. The collection of Borel sets was introduced exactly to allow these operations. We recall that a countable union of closed sets may not be closed (or open) and that a countable intersection of open sets, also, may not be open (or closed).

In this section we introduce two additional types of sets of importance in analysis, sets of type \(G_\delta\) and sets of type \(F_\sigma\). These classes form just the beginning of the large class of Borel sets. We shall find that they are precisely the right classes of sets to solve some fundamental questions about real functions.

#### 6.6.1 Sets of Type \(G_\delta\)

\footnote{Advanced section. May be omitted.}

Recall that the union of a collection of open sets is open (regardless of how many sets are in the collection), but the intersection of a collection of open sets need not be open if the collection has infinitely many sets. For example,

\[
\bigcap_{n=1}^{\infty} \left( -\frac{1}{n}, \frac{1}{n} \right) = \{0\}.
\]

Similarly, if \(q_1, q_2, q_3, \ldots\) is an enumeration of \(\mathbb{Q}\), then

\[
\bigcap_{k=1}^{\infty} (\mathbb{R} \setminus \{q_k\}) = \mathbb{R} \setminus \mathbb{Q},
\]

the set of irrational numbers. The set \(\{0\}\) is closed (not open), and \(\mathbb{R} \setminus \mathbb{Q}\) is neither open nor closed. The
set \( \mathbb{R} \setminus \mathbb{Q} \) is a countable intersection of open sets. Such sets are of sufficient importance to give them a name.

**Definition 6.14:** A subset \( H \) of \( \mathbb{R} \) is said to be of type \( \mathcal{G}_\delta \) (or a \( \mathcal{G}_\delta \) set) if it can be expressed as a countable intersection of open sets, that is, if there exist open sets \( G_1, G_2, G_3, \ldots \) such that \( H = \bigcap_{k=1}^{\infty} G_k \).

**Example 6.15:** A closed interval \([a,b]\) or a half-open interval \((a,b]\) is of type \( \mathcal{G}_\delta \) since

\[
[a,b] = \bigcap_{n=1}^{\infty} \left( a - \frac{1}{n}, b + \frac{1}{n} \right)
\]

and

\[
(a,b] = \bigcap_{n=1}^{\infty} \left( a, b + \frac{1}{n} \right).
\]

**Theorem 6.16:** Every open set and every closed set in \( \mathbb{R} \) is of type \( \mathcal{G}_\delta \).

**Proof.** Let \( G \) be an open set in \( \mathbb{R} \). It is clear that \( G \) is of type \( \mathcal{G}_\delta \). We also show that \( G \) can be expressed as a countable union of closed sets. Express \( G \) in the form

\[
G = \bigcup_{k=1}^{\infty} (a_k, b_k)
\]

where the intervals \((a_k, b_k)\) are pairwise disjoint. Now for each \( k \in \mathbb{N} \) there exist sequences \( \{c_{kj}\} \) and \( \{d_{kj}\} \) such that the sequence \( \{c_{kj}\} \) decreases to \( a_k \), the sequence \( \{d_{kj}\} \) increases to \( b_k \) and \( c_{kj} < d_{kj} \) for each \( j \in \mathbb{N} \). Thus

\[
(a_k, b_k) = \bigcup_{j=1}^{\infty} [c_{kj}, d_{kj}].
\]
We have expressed each component interval of $G$ as a countable union of closed sets. It follows that

$$G = \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{\infty} [c_{k,j}, d_{k,j}] = \bigcup_{j,k=1}^{\infty} [c_{k,j}, d_{k,j}]$$

is also a countable union of closed sets. Now take complements. This shows that $\mathbb{R} \setminus G$ can be expressed as a countable intersection of open sets (by using the de Morgan laws). Since every closed set $F$ can be written

$$F = \mathbb{R} \setminus G$$

for some open set $G$, we have shown that any closed set is of type $G_\delta$. ■

We observed in Section 6.4 that a dense set can be small in the sense of category. For example, $\mathbb{Q}$ is a first category set. Our next result shows that a dense set of type $G_\delta$ must be large in the sense of category.

**Theorem 6.17:** Let $H$ be of type $G_\delta$ and be dense in $\mathbb{R}$. Then $H$ is residual.

*Proof.* Write

$$H = \bigcap_{k=1}^{\infty} G_k$$

with each of the sets $G_k$ open. Since $H$ is dense by hypothesis and $H \subset G_k$ for each $k \in \mathbb{N}$, each of the open sets $G_k$ is also dense. Thus $\mathbb{R} \setminus G_k$ is nowhere dense for every $k \in \mathbb{N}$, and so each $G_k$ is residual. The result now follows from Lemma 6.10. ■

**Exercises**

6.6.1 Which of the following sets are of type $G_\delta$?

(a) $\mathbb{N}$

(b) $\left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$
(c) The set \( \{ C_n : n \in \mathbb{N} \} \) of midpoints of intervals complementary to the Cantor set
(d) A finite union of intervals (that need not be open or closed)

6.6.2 Prove Theorem 6.17 for the interval \([a, b]\) in place of \(\mathbb{R}\).

6.6.3 Prove that a set \( E \) of type \( \mathcal{G}_\delta \) in \( \mathbb{R} \) is either residual or else there is an interval containing no points of \( E \).

6.6.2 Sets of Type \( F_\sigma \)

\[ \Rightarrow \text{Advanced section. May be omitted.} \]

Just as the countable intersections of open sets form a larger class of sets, the \( \mathcal{G}_\delta \) sets, so also the countable unions of closed sets form a larger class of sets.

The complements of open sets are closed. By dealing with complements of \( \mathcal{G}_\delta \) sets we arrive at the dual notion of a set of type \( F_\sigma \).

**Definition 6.18:** A subset \( E \) of \( \mathbb{R} \) is said to be of type \( F_\sigma \) (or an \( F_\sigma \) set) if it can be expressed as a countable union of closed sets; that is, if there exist closed sets \( F_1, F_2, F_3, \ldots \) such that \( E = \bigcup_{k=1}^{\infty} F_k \).

Using the de Morgan laws, we verify easily that the complement of a \( \mathcal{G}_\delta \) set is an \( F_\sigma \) and vice versa (Exercise 6.6.4). This is closely related to the fact that a set is open if and only if its complement is closed.

**Example 6.19:** The set of rational numbers, \( \mathbb{Q} \) is a set of type \( F_\sigma \). This is clear since it can be expressed as
\[
\mathbb{Q} = \bigcup_{n=1}^{\infty} \{ r_n \}
\]
where \( \{ r_n \} \) is any enumeration of the rationals. The singleton sets \( \{ r_n \} \) are clearly closed. But note that \( \mathbb{Q} \) is not of type \( \mathcal{G}_\delta \) also. It follows from Theorem 6.17 that a dense set of type \( \mathcal{G}_\delta \) must be uncountable (because a countable set is first category). In particular, \( \mathbb{Q} \) is not of type \( \mathcal{G}_\delta \) (and therefore \( \mathbb{R} \setminus \mathbb{Q} \) is not of type \( F_\sigma \)).

**Theorem 6.20:** A set is of type \( \mathcal{G}_\delta \) if and only if its complement is of type \( F_\sigma \).
Example 6.21: A half-open interval $(a, b]$ is both of type $G_\delta$ and of type $F_\sigma$:

$$(a, b] = \bigcap_{n=1}^{\infty} (a, b + \frac{1}{n}) = \bigcup_{n=1}^{\infty} \left[ a + \frac{b-a}{n}, b \right].$$

\[\blacktriangleleft\]

Note. The only subsets of $\mathbb{R}$ that are both open and closed are the empty set and $\mathbb{R}$ itself. There are, however, many sets that are of type $G_\delta$ and also of type $F_\sigma$. See Exercise 6.6.1.

We can now enlarge on Theorem 6.16. There we showed that all open sets and all closed sets are in the class $G_\delta$. We now show they are also in the class $F_\sigma$.

**Theorem 6.22:** Every open set and every closed set in $\mathbb{R}$ is both of type $F_\sigma$ and $G_\delta$.

**Proof.** In the proof of Theorem 6.16 we showed explicitly how to express any open set as an $F_\sigma$. Thus open sets are of type $F_\sigma$ as well as of type $G_\delta$ (the latter being trivial). The part pertaining to closed sets now follows by considering complements and using the de Morgan laws. The complement of a closed set is open and therefore the complement of an $F_\sigma$ set is a $G_\delta$ set. $\blacksquare$

**Exercises**

6.6.4 Verify that a subset $A$ of $\mathbb{R}$ is an $F_\sigma$ ($G_\delta$) if and only if $\mathbb{R} \setminus A$ is a $G_\delta$ ($F_\sigma$).

6.6.5 Which of the following sets are of type $F_\sigma$?

(a) $\mathbb{N}$
(b) $\left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$
(c) The set $\{C_n : n \in \mathbb{N}\}$ of midpoints of intervals complementary to the Cantor set
(d) A finite union of intervals (that need not be open or closed)
6.6.6 Prove that a set of type $F_\sigma$ in $\mathbb{R}$ is either first category or contains an open interval.

6.6.7 Let $\{f_n\}$ be a sequence of real functions defined on $\mathbb{R}$ and suppose that $f_n(x) \to f(x)$ at every point $x$. Show that

$$\{x : f(x) > \alpha\} = \bigcup_{m=1}^{\infty} \bigcap_{r=1}^{\infty} \{x : f_n(x) \geq \alpha + 1/m\}.$$ 

If each function $f_n$ is continuous, what can you assert about the set $\{x : f(x) > \alpha\}$?


6.7 Oscillation and Continuity

$\Rightarrow$ Advanced section. May be omitted.

In this section we return to a problem that we began investigating in Section 5.9 about the nature of the set of discontinuity points of a function. To discuss this set we shall need the notions of $F_\sigma$ and $G_\delta$ sets and we need to introduce a new tool, the oscillation of a function.

We begin with an example of a function $f$ that is discontinuous at every rational number and continuous at every irrational number.

**Example 6.23:** Let $q_1, q_2, q_3, \ldots$ be an enumeration of $\mathbb{Q}$. Define a function $f$ by

$$f(x) = \begin{cases} \frac{1}{k}, & \text{if } x = q_k \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Since $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$, $f$ can be continuous at a point $x$ only if $f(x) = 0$; that is, only if $x \in \mathbb{R} \setminus \mathbb{Q}$. Thus $f$ is discontinuous at every $x \in \mathbb{Q}$. To check that $f$ is continuous at each point of $\mathbb{R} \setminus \mathbb{Q}$, let $x_0 \in \mathbb{R} \setminus \mathbb{Q}$ and let $\varepsilon > 0$. Choose $k \in \mathbb{N}$ such that $1/k < \varepsilon$. Since the set $q_1, q_2, \ldots, q_k$ is a finite set not containing $x_0$, there exists $\delta > 0$ such that $|q_i - x_0| \geq \delta$ for each $i = 1, \ldots, k$. Thus if $x \in \mathbb{R}$ and $|x - x_0| < \delta$, then either $x \in \mathbb{R} \setminus \mathbb{Q}$ or $x = q_j$ for some $j > k$. In either case $|f(x) - f(x_0)| \leq \frac{1}{k} < \varepsilon$. This verifies the continuity of $f$ at $x_0$. Since $x_0$ was an arbitrary irrational point, we see that $f$ is continuous at every irrational.

\[\square\]
Our example shows that it is possible for a function to be continuous at every irrational number and discontinuous at every rational number. Is it possible for the opposite to occur? Does there exist a function \( f \) continuous on \( \mathbb{Q} \) and discontinuous on \( \mathbb{R} \setminus \mathbb{Q} \)? More generally, what sets can be the set of points of continuity of some function \( f \) defined on an interval.

We answer this question in this section. The principal tool is that of oscillation of a function at a point.

6.7.1 Oscillation of a Function

\( \Rightarrow \) Advanced section. May be omitted.

In order to describe a point of discontinuity we need a way of measuring that discontinuity. For monotonic functions the jump was used previously for such a measure. For general, nonmonotonic, functions a different tool is used.

**Definition 6.24:** Let \( f \) be defined on a nondegenerate interval \( I \). We define the oscillation of \( f \) on \( I \) as the quantity

\[
\omega_f(I) = \sup_{x, y \in I} |f(x) - f(y)|.
\]

Let’s see how oscillation relates to continuity. Suppose \( f \) is defined in a neighborhood of \( x_0 \), and \( f \) is continuous at \( x_0 \). Then

\[
\inf_{\delta>0} \omega_f((x_0 - \delta, x_0 + \delta)) = 0.
\]

To see this, let \( \varepsilon > 0 \). Since \( f \) is continuous at \( x_0 \), there exists \( \delta_0 > 0 \) such that

\[
|f(x) - f(x_0)| < \varepsilon/2
\]

if \( |x - x_0| < \delta_0 \). If

\[
x_0 - \delta_0 < x_1 < x_2 < x_0 + \delta_0,
\]

then

\[
|f(x_1) - f(x_2)| \leq |f(x_1) - f(x_0)| + |f(x_0) - f(x_2)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
Since (2) is valid for all \(x_1, x_2 \in (x_0 - \delta_0, x_0 + \delta_0)\), we have
\[
\sup \{|f(x_1) - f(x_2)| : x_0 - \delta_0 < x_1 \leq x_2 < x_0 + \delta_0\} \leq \varepsilon.
\] (3)
But (3) implies that if \(0 < \delta < \delta_0\), then
\[
\omega_f([x_0 - \delta, x_0 + \delta]) \leq \varepsilon.
\]
Since \(\varepsilon\) was arbitrary, the result follows.

The converse is also valid. Suppose (1) holds. Let \(\varepsilon > 0\). Choose \(\delta > 0\) such that
\[
\omega_f(x_0 - \delta, x_0 + \delta) < \varepsilon.
\]
Then
\[
\sup \{|f(x) - f(x_0)| : x \in (x_0 - \delta, x_0 + \delta)\} < \varepsilon,
\]
so \(|f(x) - f(x_0)| < \varepsilon\) whenever \(|x - x_0| < \delta\). This implies continuity of \(f\) at \(x_0\).

We summarize the preceding as a theorem.

**Theorem 6.25:** Let \(f\) be defined on an interval \(I\) and let \(x_0 \in I\). Then \(f\) is continuous at \(x_0\) if and only if
\[
\inf_{\delta > 0} \omega_f((x_0 - \delta, x_0 + \delta)) = 0.
\]

The quantity in the statement of the theorem is sufficiently important to have a name.

**Definition 6.26:** Let \(f\) be defined in a neighborhood of \(x_0\). The quantity
\[
\omega_f(x_0) = \inf_{\delta > 0} \omega_f((x_0 - \delta, x_0 + \delta))
\]
is called the oscilation of \(f\) at \(x_0\).

Theorem 6.25 thus states that a function \(f\) is continuous at a point \(x_0\) if and only if \(\omega_f(x_0) = 0\).
Returning to the function that introduced this section, we see that
\[
\omega_f(x) = \begin{cases} 
1/k, & \text{if } x = q_k \\
0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}.
\end{cases}
\]

Let’s now see how the concept of oscillation relates to the set of points of continuity of a function.

**Theorem 6.27:** Let \( f \) be defined on a closed interval \( I \) (which may be all of \( \mathbb{R} \)). Let \( \gamma > 0 \). Then the set
\[ \{ x : \omega_f(x) < \gamma \} \]
is open and the set
\[ \{ x : \omega_f(x) \geq \gamma \} \]
is closed.

**Proof.** Let \( A = \{ x : \omega_f(x) < \gamma \} \) and let \( x_0 \in A \). We wish to find a neighborhood \( U \) of \( x_0 \) such that \( U \subset A \); that is, such that \( \omega_f(x) < \gamma \) for all \( x \in U \).

Let \( \omega_f(x_0) = \alpha < \gamma \) and let \( \beta \in (\alpha, \gamma) \). From Definition 6.26 we infer the existence of a number \( \delta > 0 \) such that
\[ |f(u) - f(v)| \leq \beta \]
for \( u, v \in (x_0 - \delta, x_0 + \delta) \). Let
\[ U = (x_0 - \delta, x_0 + \delta) \]
and let \( x \in U \). Since \( U \) is open, there exists \( \delta_1 < \delta \) such that
\[ (x - \delta_1, x + \delta_1) \subset U. \]

Then
\[
\omega_f(x) \leq \sup \{|f(t) - f(s)| : t, s \in (x - \delta_1, x + \delta_1)\} \\
\leq \sup \{|f(u) - f(v)| : u, v \in U\} \leq \beta < \gamma,
\]
so \( x \in A \). This proves \( A \) is open. It follows then that the complement of \( A \) in \( I \), the set

\[
\{ x : \omega_f(x) \geq \gamma \},
\]

must be closed.

We use the oscillation in the next subsection to answer a question about the nature of the set of points of continuity of a function.

**Exercises**

6.7.1 Suppose that \( f \) is bounded on an interval \( I \). Prove that

\[
\omega_f(I) = \sup_{x \in I} f(x) - \inf_{x \in I} f(x).
\]

6.7.2 A careless student believes that the oscillation can be written as

\[
\omega_f(x_0) = \limsup_{x \to x_0} f(x) - \liminf_{x \to x_0} f(x).
\]

Show that this is not true, even for bounded functions.

6.7.3 Prove that

\[
\omega_f(x_0) = \lim_{\delta \to 0^+} \omega_f((x_0 - \delta, x_0 + \delta)).
\]

See Note 159

6.7.4 Calculate \( \omega_f(0) \) for each of the following functions.

(a) \( f(x) = \begin{cases} x, & \text{if } x \neq 0 \\ 4, & \text{if } x = 0 \end{cases} \)

(b) \( f(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Q} \\ 1, & \text{if } x \notin \mathbb{Q} \end{cases} \)

(c) \( f(x) = \begin{cases} n, & \text{if } x = \frac{1}{n} \\ 0, & \text{otherwise} \end{cases} \)
(d) \[ f(x) = \begin{cases} \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \]

(e) \[ f(x) = \begin{cases} \sin \frac{1}{x}, & \text{if } x \neq 0 \\ \frac{7}{x}, & \text{if } x = 0 \end{cases} \]

(f) \[ f(x) = \begin{cases} \frac{1}{x} \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \]

6.7.5 In the proof of Theorem 6.27 we let \( \omega_f(x_0) = \alpha < \gamma \) and let \( \beta \in (\alpha, \gamma) \). Why was the \( \beta \) introduced? Would the proof have worked if we had used \( \beta = \gamma \)?

### 6.7.2 The Set of Continuity Points

*Advanced section. May be omitted.*

Given an arbitrary function, how can we describe the nature of the set of points where \( f \) is continuous? Can it be any set? Given a set \( E \), how can we know whether there is a function that is continuous at every point of \( E \) and discontinuous at every point not in \( E \)?

We saw in Example 6.23 that a function exists whose set of continuity points is exactly the irrationals. Can a function exist whose set of continuity points is exactly the rationals? By characterizing the set of such points we can answer this and other questions about the structure of functions.

We now prove the main result of this section using primarily the notion of oscillation introduced in Section 6.7.1.

**Theorem 6.28:** Let \( f \) be defined on a closed interval \( I \) (which may be all of \( \mathbb{R} \)). Then the set \( C_f \) of points of continuity of \( f \) is of type \( G_\delta \), and the set \( D_f \) of points of discontinuity of \( f \) is of type \( F_\sigma \). Conversely, if \( H \) is a set of type \( G_\delta \), then there exists a function \( f \) defined on \( \mathbb{R} \) such that \( C_f = H \).

**Proof.** To prove the first part, let \( f : I \to \mathbb{R} \). We show that the set

\[ C_f = \{ x : \omega_f(x) = 0 \} \]

is of type \( G_\delta \). For each \( k \in \mathbb{N} \), let

\[ B_k = \left\{ x : \omega_f(x) \geq \frac{1}{k} \right\} . \]
By Theorem 6.27, each of the sets $B_k$ is closed. Thus the set

$$B = \bigcup_{k=1}^{\infty} B_k$$

is of type $\mathcal{F}_\sigma$. By Theorem 6.25, $D_f = B$. Therefore, $C_f = I \setminus B$. Since the complement of an $\mathcal{F}_\sigma$ is a $\mathcal{G}_\delta$, the set $C_f$ is a $\mathcal{G}_\delta$.

To prove the converse, let $H$ be any subset of $\mathbb{R}$ of type $\mathcal{G}_\delta$. Then $H$ can be expressed in the form

$$H = \bigcap_{k=1}^{\infty} G_k$$

with each of the sets $G_k$ being open. We may assume without loss of generality that $G_1 = \mathbb{R}$ and that $G_i \supset G_{i+1}$ for each $i \in \mathbb{N}$. (Verify this.)

Let $\{\alpha_k\}$ and $\{\beta_k\}$ be sequences of positive numbers, each converging to zero, with

$$\alpha_k > \beta_k > \alpha_{k+1},$$

for all $k \in \mathbb{N}$. Define a function $f: \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 
0 & \text{if } x \in H \\
\alpha_k & \text{if } x \in (G_k \setminus G_{k+1}) \cap \mathbb{Q} \\
\beta_k & \text{if } x \in (G_k \setminus G_{k+1}) \cap (\mathbb{R} \setminus \mathbb{Q}).
\end{cases}$$

We show that $f$ is continuous at each point of $H$ and discontinuous at each point of $\mathbb{R} \setminus H$.

Let $x_0 \in H$ and let $\varepsilon > 0$. Choose $n$ such that $\alpha_n < \varepsilon$. Since

$$x_0 \in H = \bigcap_{k=1}^{\infty} G_k,$$

we see that $x_0 \in G_n$. The set $G_n$ is open, so there exists $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset G_n$. From the
definition of \( f \) on \( G_n \), we see that

\[
0 \leq f(x) \leq \alpha_n < \varepsilon
\]

for all \( x \in (x_0 - \delta, x_0 + \delta) \). Thus

\[
|f(x) - f(x_0)| = |f(x) - 0| = |f(x)| < \varepsilon
\]

if \( |x - x_0| < \delta \), so \( f \) is continuous at \( x_0 \).

Now let \( x_0 \in \mathbb{R} \setminus H \). Then there exists \( k \in \mathbb{N} \) such that \( x_0 \) belongs to the set \( G_k \setminus G_{k+1} \). Thus \( f(x_0) = \alpha_k \) or \( f(x_0) = \beta_k \). Let us suppose that \( f(x_0) = \alpha_k \). If \( x_0 \) is an interior point of \( G_k \setminus G_{k+1} \), then \( x_0 \) is a limit point of

\[
\{ x : x \in (G_k \setminus G_{k+1}) \cap (\mathbb{R} \setminus \mathbb{Q}) \} = \{ x : f(x) = \beta_k \},
\]

so \( f \) is discontinuous at \( x_0 \).

The argument is similar if \( x_0 \) is a boundary point of \( G_k \setminus G_{k+1} \). Again, assume \( f(x_0) = \alpha_k \). Arbitrarily close to \( x_0 \) there are points of the set

\[
\mathbb{R} \setminus (G_k \setminus G_{k+1}).
\]

At these points, \( f \) takes on values in the set

\[
S = \{ 0 \} \cup \bigcup_{i \neq k} \alpha_i \cup \bigcup_{j \neq k} \beta_j.
\]

The only limit point of this set is zero and so \( S \) is closed. In particular, \( \alpha_k \) is \emph{not} a limit point of this set and does not belong to the set. Let \( \varepsilon \) be half the distance from the point \( \alpha_k \) to the closed set \( S \); that is, let

\[
\varepsilon = \frac{1}{2} d(\alpha_k, S).
\]

Arbitrarily close to \( x_0 \) there are points \( x \) such that \( f(x) \in S \). For such a point,

\[
|f(x) - f(x_0)| = |f(x) - \alpha_k| > \varepsilon,
\]

so \( f \) is discontinuous at \( x_0 \).
Observe that Theorem 6.28 answers a question we asked earlier: Is there a function \( f \) continuous on \( \mathbb{Q} \) and discontinuous at every point of \( \mathbb{R} \setminus \mathbb{Q} \)? The answer is negative, since \( \mathbb{Q} \) is not of type \( G_{\delta} \).

**Exercises**

6.7.6 In the second part of the proof of Theorem 6.28 we provided a construction for a function \( f \) with \( C_f = H \), where \( H \) is an arbitrary set of type \( G_{\delta} \). Exhibit explicitly sets \( G_k \) that will give rise to a function \( f \) such that \( C_f = \mathbb{R} \setminus \mathbb{Q} \). Can you do this in such a way that the resulting function is the one we obtained at the beginning of this section?

6.7.7 In the proof of Theorem 6.28 we took \( \varepsilon = \frac{1}{2} d(\alpha_k, S) \). Show that this number equals

\[
\frac{1}{2} \min \{ \min \{|\alpha_i - \alpha_k|, |\beta_i - \beta_k|\} \}.
\]

6.8 **Sets of Measure Zero**

*Advanced section. May be omitted.*

In analysis there are a number of ways in which a set might be considered as “small.” For example, the Cantor set is not small in the sense of counting: It is uncountable. It is small in another different sense: It is nowhere dense, that is there is no interval at all in which it is dense. Now we turn to another way in which the Cantor set can be considered small: It has “zero length.”

**Example 6.29:** Suppose we wish to measure the “length” of the Cantor set. Since the Cantor set is rather bizarre, we might look instead at the sequence of intervals that have been removed. There is no difficulty in assigning a meaning of length to an interval; the length of \((a, b)\) is \( b - a \). What is the total length of the intervals removed in the construction of the Cantor set? From the interval \([0, 1]\) we remove first a middle third interval of length \( 1/3 \), then two middle third intervals of length \( 1/9 \), and so on so that at the \( n \)th stage we remove \( 2^{n-1} \) intervals each of length \( 3^{-n} \). The sum of the lengths of all intervals so removed is

\[
\frac{1}{3} + 2(1/9) + 4(1/27) + \cdots =
\]

\[
\frac{1}{3} \left( 1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \ldots \right) = 1.
\]
From the interval $[0, 1]$ we appear to have removed all of the length. What is left over, the Cantor set, must have length zero.

This method of computing lengths has some merit but it is not the one we wish to adopt here. Another approach to “measuring” the length of the Cantor set is to consider the length that remains at each stage. At the first stage the Cantor set is contained inside the union

$$[0, 1/3] \cup [2/3, 1],$$

which has length $2(1/3)$. At the next stage it is contained inside a union of four intervals, with total length $4(1/9)$. Similarly, at the $n$th stage the Cantor set is contained inside the union of $2^n$ intervals each of length $3^{-n}$. The sum of the lengths of all these intervals is $(2/3)^n$, and this tends to zero as $n$ gets large. Thus, as before, it seems we should assign zero length to the Cantor set.

We convert the second method of the example into a definition of what it means for a set to be of measure zero. “Measure” is the technical term used to describe the “length” of sets that need not be intervals. In the example we used closed intervals while in our definition we have employed open intervals. There is no difference (see Exercise 6.8.13). In the example we covered the Cantor set with a finite sequence of intervals while in our definition we have employed an infinite sequence. For the Cantor set there is no difference but for other sets (sets that are not bounded or are not closed) there is a difference.

**Definition 6.30:** Let $E$ be a set of real numbers. Then $E$ is said to have measure zero if for every $\varepsilon > 0$ there is a finite or infinite sequence

$$(a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4), \ldots$$

of open intervals covering the set $E$ so that

$$\sum_{k=1}^{\infty} (b_k - a_k) \leq \varepsilon.$$

**Note.** In the definition of measure zero sets is there a change if we insist on an infinite sequence of
intervals, disallowing finite sequences? Suppose that the sequence
\[(a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4), \ldots (a_N, b_N)\]
of open intervals covers the set \(E\) so that
\[
\sum_{k=1}^{N} (b_k - a_k) < \varepsilon/2.
\]
Then to satisfy the definition we could add in some further intervals that do not amount in length to more than \(\varepsilon/2\). For example, take
\[(a_{N+p}, b_{N+p}) = (0, \varepsilon/2^{p+1})\]
for \(p = 1, 2, 3, \ldots\). Then
\[
\sum_{k=1}^{\infty} (b_k - a_k) = \sum_{k=1}^{N} (b_k - a_k) + \sum_{p=1}^{\infty} \varepsilon/2^{p+1} < \varepsilon.
\]
Thus the definition would not be changed if we had required infinite coverings.

Here are some examples of sets of measure zero.

**Example 6.31:** Every finite set has measure zero. The empty set is easily handled. If \(E = \{x_1, x_2, \ldots, x_N\}\) and \(\varepsilon > 0\), then the sequence of intervals
\[
\left(x_i - \frac{\varepsilon}{2N}, x_i + \frac{\varepsilon}{2N}\right) \quad i = 1, 2, 3, \ldots, N
\]
covers the set \(E\) and the sum of all the lengths is \(\varepsilon\).

**Example 6.32:** Every infinite, countable set has measure zero. If \(E = \{x_1, x_2, \ldots\}\)
and $\varepsilon > 0$, then the sequence of intervals

$$(x_i - \frac{\varepsilon}{2^{i+1}}, x_i + \frac{\varepsilon}{2^{i+1}}) \quad i = 1, 2, 3, \ldots$$

covers the set $E$ and, since

$$\sum_{k=1}^{\infty} 2 \left( \frac{\varepsilon}{2^{k+1}} \right) = \sum_{k=1}^{\infty} \varepsilon 2^{-k} = \varepsilon,$$

the sum of all the lengths is $\varepsilon$.

**Example 6.33:** The Cantor set has measure zero. Let $\varepsilon > 0$. Choose $n$ so that $(2/3)^n < \varepsilon$. Then the $n$th stage intervals in the construction of the Cantor set give us $2^n$ closed intervals each of length $(1/3)^n$. This covers the Cantor set with $2^n$ closed intervals of total length $(2/3)^n$, which is less than $\varepsilon$. If the closed intervals trouble you (the definition requires open intervals), see Exercise 6.8.13 or argue as follows. Since $(2/3)^n < \varepsilon$ there is a positive number $\delta$ so that

$$(2/3)^n + \delta < \varepsilon.$$  

Enlarge each of the closed intervals to form a slightly larger open interval, but change the length of each only enough so that the sum of the lengths of all the $2^n$ closed intervals does not increase by more than $\delta$. The resulting collection of open intervals also covers the Cantor set, and the sum of the length of these intervals is less than $\varepsilon$.

One of the most fundamental of the properties of sets having measure zero is how sequences of such sets combine. We recall that the union of any sequence of countable sets is also countable. We now prove that the union of any sequence of measure zero sets is also a measure zero set.

**Theorem 6.34:** Let $E_1, E_2, E_3, \ldots$ be a sequence of sets of measure zero. Then the set $E$ formed by taking the union of all the sets in the sequence is also of measure zero.
Proof. Let \( \varepsilon > 0 \). We shall construct a cover of \( E \) consisting of a sequence of open intervals of total length less than \( \varepsilon \). Since \( E_1 \) has measure zero, there is a sequence of open intervals 
\[(a_{11}, b_{11}), (a_{12}, b_{12}), (a_{13}, b_{13}), (a_{14}, b_{14}), \ldots \]
covering the set \( E_1 \) and so that the sum of the lengths of these intervals is smaller than \( \varepsilon /2 \). Since \( E_2 \) has measure zero, there is a sequence of open intervals 
\[(a_{21}, b_{21}), (a_{22}, b_{22}), (a_{23}, b_{23}), (a_{24}, b_{24}), \ldots \]
covering the set \( E_2 \) and so that the sum of the lengths of these intervals is smaller than \( \varepsilon /4 \). In general, for each \( k = 1, 2, 3, \ldots \) there is a sequence of open intervals 
\[(a_{k1}, b_{k1}), (a_{k2}, b_{k2}), (a_{k3}, b_{k3}), (a_{k4}, b_{k4}), \ldots \]
covering the set \( E_k \) and so that the sum of the lengths of these intervals is smaller than \( \varepsilon /2^k \). The totality of all these intervals can be arranged into a single sequence of open intervals that covers every point in the union of the sequence \( \{E_k\} \). The sum of the lengths of all the intervals in the large sequence is smaller than 
\[\varepsilon /2 + \varepsilon /4 + \varepsilon /8 + \cdots = \varepsilon.\]
It follows that \( E \) has measure zero.

Let us return to the situation for the Cantor set once again. For each \( \varepsilon > 0 \) we were able to choose a finite cover of open intervals with total length less than \( \varepsilon \). This is not the case for all sets of measure zero. For example, the set of all rational numbers on the real line is countable and hence also of measure zero. Any finite collection of intervals must fail to cover that set, in fact cannot come close to covering all rational numbers. For what sets is it possible to select finite coverings of small length? The answer is that this is possible for compact sets of measure zero.
**Theorem 6.35:** Let \( E \) be a compact set of measure zero. Then for every \( \varepsilon > 0 \) there is a finite collection of open intervals

\[(a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4), \ldots, (a_N, b_N)\]

that covers the set \( E \) and so that

\[\sum_{k=1}^{N} (b_k - a_k) < \varepsilon.\]

**Proof.** Since \( E \) has measure zero, it is certainly possible to select a sequence of open intervals

\[(a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4), \ldots\]

that covers the set \( E \) and so that

\[\sum_{k=1}^{\infty} (b_k - a_k) < \varepsilon.\]

But how can we reduce this collection to a finite one that also covers the set \( E \)? If you studied the Heine-Borel theorem (Theorem 4.33), then you know how.

We shall present here a proof that uses the Bolzano-Weierstrass theorem instead. We claim that we can find an integer \( N \) so that all points of \( E \) are in one of the intervals

\[(a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4), \ldots, (a_N, b_N).\]

This will prove the theorem.

We prove this by contradiction. If this is not so, then for each integer \( k = 1, 2, 3, \ldots \) we must be able to find a point \( x_k \in E \) but \( x_k \) is not in any of the intervals

\[(a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4), \ldots, (a_k, b_k).\]

The sequence \( \{x_k\} \) is bounded because \( E \) is bounded. By the Bolzano-Weierstrass theorem the sequence has a convergent subsequence \( \{x_{n_j}\} \). Let \( z \) be the limit of the convergent subsequence. Since \( E \) is closed \( z \) is in \( E \). The original sequence of intervals covers all of \( E \) and so there must be an interval \((a_M, b_M)\) that
contains \( z \). For large values of \( j \) the points \( x_{n_j} \) also belong to \((a_M, b_M)\). But this is impossible since \( x_{n_j} \) cannot belong to the interval \((a_M, b_M)\) for \( n_j \geq M \). Since this is a contradiction, the proof is done.  

Exercises

6.8.1 Show that every subset of a set of measure zero also has measure zero.

6.8.2 If \( E \) has measure zero, show that the translated set

\[
E + \alpha = \{ x + \alpha : x \in E \}
\]

also has measure zero.

6.8.3 If \( E \) has measure zero, show that the expanded set

\[
eE = \{ cx : x \in E \}
\]

also has measure zero for any \( c > 0 \).

6.8.4 If \( E \) has measure zero, show that the reflected set

\[
-E = \{ -x : x \in E \}
\]

also has measure zero.

6.8.5 Without referring to the proof of Theorem 6.34, show that the union of any two sets of measure zero also has measure zero.

6.8.6 If \( E_1 \subset E_2 \) and \( E_1 \) has measure zero but \( E_2 \) has not, what can you say about the set \( E_2 \setminus E_1 \)?

6.8.7 Show that any interval \((a, b)\) or \([a, b]\) is not of measure zero.

6.8.8 Give an example of a set that is not of measure zero and does not contain any interval \([a, b]\).

6.8.9 A careless student claims that if a set \( E \) has measure zero, then it is obviously true that the closure \( \overline{E} \) must also have measure zero. Is this correct?

6.8.10 If a set \( E \) has measure zero what can you say about interior points of that set?

6.8.11 Suppose that a set \( E \) has the property that \( E \cap [a, b] \) has measure zero for every compact interval \([a, b]\). Must \( E \) also have measure zero?
6.8.12 Show that the set of real numbers in the interval [0, 1] that do not have a 7 in their infinite decimal expansion is of measure zero.

6.8.13 In Definition 6.30 show that closed intervals may be used without changing the definition.

6.8.14 Describe completely the class of sets \( E \) with the following property: For every \( \varepsilon > 0 \) there is a finite collection of open intervals 
\[(a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4), \ldots, (a_N, b_N)\]
that covers the set \( E \) and so that
\[
\sum_{k=1}^{N} (b_k - a_k) < \varepsilon.
\]
(These sets are said to have zero content.)

6.8.15 Show that a set \( E \) has measure zero if and only if there is a sequence of intervals 
\[(a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4), \ldots\]
so that every point in \( E \) belongs to infinitely many of the intervals and \( \sum_{k=1}^{\infty} (b_k - a_k) \) converges.

6.8.16 By altering the construction of the Cantor set, construct a nowhere dense closed subset of [0, 1] so that the sum of the lengths of the intervals removed is not equal to 1. Will this set have measure zero?

6.9 Challenging Problems for Chapter 6

6.9.1 Show that a function is discontinuous except at the points of a first category set if and only if it is continuous at a dense set of points.

6.9.2 Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function. Assume that for every positive number \( \varepsilon \) the sequence \( \{f(n\varepsilon)\} \) converges to zero as \( n \to \infty \). Prove that
\[
\lim_{x \to \infty} f(x) = 0.
\]
6.9.3 Let \( f_n \) be a sequence of continuous functions defined on an interval \([a, b]\) such that \( \lim_{n \to \infty} f_n(x) = 0 \) for each \( x \in [a, b] \). Show that for any \( \varepsilon > 0 \) there is an interval \([c, d]\) \( \subset [a, b] \) and an integer \( N \) so that
\[
|f_n(x)| < \varepsilon
\]
for every \( n \geq N \) and every \( x \in [c, d] \). Show that this need not be true for \([c, d] = [a, b]\).

6.9.4 Let \( f_n \) be a sequence of continuous functions defined on an interval \([a, b]\) such that \( \lim_{n \to \infty} f_n(x) = \infty \) for each \( x \in [a, b] \). Show that for any \( M > 0 \) there is an interval \([c, d]\) \( \subset [a, b] \) and an integer \( N \) so that
\[
f_n(x) > M
\]
for every \( n \geq N \) and every \( x \in [c, d] \). Show that this need not be true for \([c, d] = [a, b]\).

Notes

147 Exercise 6.2.9. To make this true, assume that \( f \) is onto or else show that if \( E \) is dense then \( f(E) \) is dense in the set (interval) \( f(\mathbb{R}) \).

148 Exercise 6.3.1. If \( q_1, q_2, q_3, \ldots \) is an enumeration of the rationals, then each of the sets \( \{q_i\}, i \in \mathbb{N}, \) is nowhere dense, but
\[
\bigcup_{i=1}^{n} \{q_i\} = \mathbb{Q}
\]
is not nowhere dense. (Indeed it is dense.)

149 Exercise 6.3.2. All of (a)–(e) and (h) are true. Find counterexamples for (f) and (g). The proofs that the others are true follow routinely from the definition.

150 Exercise 6.4.1. Suppose that
\[
A_n = \bigcup_{k=1}^{\infty} A_{nk}
\]
with each of the sets $A_{nk}$ nowhere dense. Then
\[ \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} A_{nk} = \bigcup_{n,k=1}^{\infty} A_{nk} \]
expresses that union as a first category set.

151 Exercise 6.4.2. Let $\{B_n\}$ be a sequence of residual subsets of $\mathbb{R}$. Thus each of the sets $B_n$ is the complement of a first category set $A_n$. For each $n$ write
\[ A_n = \bigcup_{k=1}^{\infty} A_{nk} \]
with each of the sets $A_{nk}$ nowhere dense. Then
\[ B_n = \mathbb{R} \setminus \bigcup_{k=1}^{\infty} A_{nk}. \]
Now use De Morgan’s laws.

152 Exercise 6.4.3. Suppose that $X$ is residual, that is,
\[ X = \mathbb{R} \setminus \bigcup_{n=1}^{\infty} Q_n \]
where each $Q_n$ is nowhere dense. Show that for any interval $[a, b]$ there is a point in $X \cap [a, b]$ by constructing an appropriate descending sequence of closed subintervals of $[a, b]$.

153 Exercise 6.4.4. Make sure your sets are dense but not both residual (e.g., $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$).

154 Exercise 6.4.5. This follows, with the correct interpretation, directly from the Baire category theorem.

155 Exercise 6.4.7. Consider the sequence
\[ A_N = \{ x \in [0, 1] : |f_n(x)| \leq 1, \text{ all } n \geq N \}. \]
Check that
\[ \bigcup_{N=1}^{\infty} A_N = [0, 1]. \]

156 Exercise 6.5.7. It is clear that there must be many irrational numbers in the Cantor ternary set, since that set is uncountable and the rationals are countable. Your job is to find just one.

157 Exercise 6.5.10. Consider \( G = (0, 1) \setminus C \) where \( C \) is the Cantor ternary set.

158 Exercise 6.6.7. Often to prove a set identity such as this the best way is to start with a point \( x \) that belongs to the set on the right and then show that point must be in the set on the left. After that is successful start with a point \( x \) that belongs to the set on the left. For example, if \( f(x) > \alpha \), then

\[ f(x) \geq \alpha + 1/m \]

for some integer \( m \). But

\[ f_n(x) \to f(x) \]

and so there must be an integer \( R \) so that \( f_n(x) > \alpha + 1/m \) for all \( n \geq R \), etc.

This exercise shows how unions and intersections of sequences of open and closed sets might arise in analysis. Note that the sets

\[ \{ x : f_n(x) \geq \alpha + 1/m \} \]

would be closed if the functions \( f_n \) are continuous. Thus it would follow that the set

\[ \{ x : f(x) > \alpha \} \]

must be of type \( F_\sigma \). This says something interesting about a function \( f \) that is the limit of a sequence of continuous functions \( \{ f_n \} \).

159 Exercise 6.7.3. You need to recall Theorem 5.60, which asserts that monotone functions have left- and right-hand limits.

160 Exercise 6.9.2. This is from the 1964 Putnam Mathematical Competition.