Appendix B

EXTREMA

This appendix was supplied to us by Professor Jozef Doboš[1]*. We thank him for permission to include the material in this distribution. This chapter will be a useful supplement for instructors who wish to enlarge on the material of Chapters 11 and 12.*

B.1 Local Extrema

We start by a standard task from the coordinate geometry:

Example B.1: Compute the minimum distance between two skew lines. Suppose that each line is defined by a point A and a directional vector $\boldsymbol{s}$. For example, $A_1 = (-4, 4, -1)$, $\boldsymbol{s}_1 = (2, -1, -2)$ for the first line, and $A_2 = (-5, 5, 5)$, $\boldsymbol{s}_2 = (4, -3, -5)$ for the second one. Any two points of the lines may be written as points in the form $A_1 + p\boldsymbol{s}_1$ and $A_2 + q\boldsymbol{s}_2$, where p and q are real parameters. The distance between the points

$$(-4 + 2p, 4 - p, -1 - 2p) \text{ and } (-5 + 4q, 5 - 3q, 5 - 5q)$$

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can be calculated as

$$d(p,q) = \sqrt{(1+2p-4q)^2 + (-1-p+3q)^2 + (-6-2p+5q)^2}.$$

We rewrite this expression as

$$d(p,q) = \sqrt{(3p-7q+5)^2 + (q-2)^2 + 9}.$$

It is easy to see that $d(p,q) \geq 3$ for each $p, q \in \mathbb{R}$.

Therefore the minimum of this expression is $f(3,2) = 3$. ◀

Definition B.2: Let f be a function defined on the set $D \subset \mathbb{R}^2$. Let $(x_0, y_0) \in D$. We say that f has at the point (x_0, y_0) a *local minimum*, if there is a neighborhood of the point (x_0, y_0) such that for each (x, y) from this neighborhood and that is in the set D, we have $f(x,y) \geq f(x_0, y_0)$.

A local maximum is similarly defined.

A definition of a *strict local minimum* we obtain by requiring the strict inequality $f(x,y) > f(x_0, y_0)$ in the previous definition, naturally with a restriction to $(x,y) \neq (x_0, y_0)$. A strict local maximum is similarly defined.

Theorem B.3: *If f has a local extremum at the point (x_0, y_0) and if the first partial derivatives of the function at this point exist, then they equal zero.*

Proof. Suppose f has a local extremum at the point (x_0, y_0). Then the function $f(x_0, \cdot)$ has a local extremum at the point y_0, and the function $f(\cdot, y_0)$ has a local extremum at the point x_0. Therefore the derivatives of these functions equal zero. ■

Example B.4: Consider the function $f(x,y) = ax^2 + 2bxy + cy^2$. Suppose that $a \neq 0$. Then by completing the square we obtain

$$f(x,y) = a\left(x + \frac{b}{a}y\right)^2 + \frac{1}{a}\left(ac - b^2\right)y^2.$$

If $a > 0$, $ac - b^2 > 0$, then $f(x,y)$ is a sum of two nonnegative expressions. Therefore for each $(x,y) \in \mathbb{R}^2$ we have $f(x,y) \geq 0$, where equality holds if and only if $(x,y) = (0,0)$. This shows that the function f has a strict local minimum at the point $(0,0)$. Similarly in a case $a<0$, $ac-b^2>0$ the function f has a strict local maximum at the point $(0,0)$.

Now, suppose that $a \neq 0$, $ac-b^2<0$. Then $f(x,y)$ is a sum of two expressions with opposite signs. For a suitable choice of numbers x and y it may happen that one of them equals zero and other one does not. For example

$$f(x,0)=ax^2, \qquad f\left(-\frac{b}{a}y,y\right)=\frac{1}{a}\left(ac-b^2\right)y^2.$$

Evidently in each neighborhood of the point $(0,0)$ there are points $(x,0)$ and points $\left(-\frac{b}{a}y,y\right)$, whereas values of f at those points does not equal zero and have opposite signs (for instance in the case $x=y=\frac{1}{n}$). This shows that f does not have an extremum at the point $(0,0)$.

Finally, let $ac-b^2<0$ and $a=0$. Then $b\neq 0$ and

$$f\left(\frac{1-c}{2b}x,x\right)=x^2, \quad f\left(\frac{1+c}{2b}x,-x\right)=-x^2.$$

Therefore the function f does not have an extremum at the point $(0,0)$.

This shows that:

1. If $ac-b^2>0$ and $a>0$, then f has a strict local minimum at $(0,0)$.

2. If $ac-b^2>0$ and $a<0$, then f has a strict local maximum at $(0,0)$.

3. If $ac-b^2<0$, then f does not have an extremum at the point $(0,0)$.

◀

Note, in this example, that there is a dependence between the coefficients a, b, c and second partial derivatives of the function:

$$f_{11}(0,0)=2a, \quad f_{12}(0,0)=2b, \quad f_{22}(0,0)=2c.$$

This anticipates the following assertion.

Theorem B.5: *Let f have a continuous partial derivatives of the second order on an open set D. Let $(x_0, y_0) \in D$ be a critical point of f (i.e., a point where the partial derivatives equal zero). Put*

$$a = f_{11}(x_0, y_0), \quad b = f_{12}(x_0, y_0), \quad c = f_{22}(x_0, y_0).$$

1. *If $ac - b^2 > 0$ and $a > 0$, then f has a strict local minimum at (x_0, y_0).*

2. *If $ac - b^2 > 0$ and $a < 0$, then f has a strict local maximum at (x_0, y_0).*

3. *If $ac - b^2 < 0$, then f does not have an extremum at (x_0, y_0).*

Proof. (1) From continuity of the second partial derivatives of f at the point (x_0, y_0) we obtain that there exists $\delta > 0$ such that $f_{11} > 0$ and $f_{11}f_{22} - f_{12}^2 > 0$ for every $(x, y) \in B((x_0, y_0), \delta)$. Then for the second differential of f, which is defined by

$$df^2(x, y, dx, dy) = f_{11}dx^2 + 2f_{12}dxdy + f_{22}dy^2,$$

for each $(x, y) \in B((x_0, y_0), \delta)$ and for each $(dx, dy) \neq (0, 0)$, we have

$$df^2(x, y, dx, dy) > 0.$$

Indeed,

$$df^2(x, y, dx, dy) = f_{11}\left(dx + \frac{f_{12}}{f_{11}}dy\right)^2 + \frac{1}{f_{11}}\left(f_{11}f_{22} - f_{12}^2\right)dy^2$$

is a sum of two positive expressions.

Let $(x_1, y_1) \in B((x_0, y_0), \delta)$, $(x_1, y_1) \neq (x_0, y_0)$. We show that $f(x_1, y_1) > f(x_0, y_0)$. Put

$$g(t) = f(x(t), y(t))$$

where $x(t) = x_0 + tdx$, $y(t) = y_0 + tdy$, $dx = x_1 - x_0$, $dy = y_1 - y_0$. Moreover

$$(x(t), y(t)) \in B((x_0, y_0), \delta)$$

if and only if $|t| < \delta/\sqrt{dx^2 + dy^2}$.

Therefore

$$g'(t) = f_1(x(t), y(t))dx + f_2(x(t), y(t))dy.$$

Since (x_0, y_0) is a critical point of the function f, we have $g'(0) = 0$. Further

$$g''(t) = f_{11}(x(t), y(t))u_1^2 + 2f_{12}(x(t), y(t))u_1u_2 + f_{22}(x(t), y(t))u_2^2 =$$

$$= df^2(x(t), y(t), dx, dy) > 0.$$

Since $\delta/\sqrt{dx^2 + dy^2} > 1$, applying Lagrange's Theorem (Theorem 7.43) we obtain

$$g(1) = g(0) + g'(0) + \frac{1}{2}g''(t_1),$$

where $0 \le t_1 \le 1$. Since $g'(0) = 0$ a $g''(t_1) > 0$, we have $f(x_1, y_1) = g(1) > g(0) = f(x_0, y_0)$.

The case (2) is similar.

Finally, we prove the case (3). We find two unit vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ such that the derivatives $g''_{\boldsymbol{u}}(0)$ and $g''_{\boldsymbol{v}}(0)$ have opposite signs and they does not equal zero, where $g_{\boldsymbol{w}}(t) = f(x_0 + tw_1, y_0 + tw_2)$ and $\boldsymbol{w} = (w_1, w_2)$ is a unit vector.

Suppose that $a = c = 0$. Evidently $b \neq 0$. Put $\boldsymbol{u} = (u_1, u_2)$ and $\boldsymbol{v} = (-u_1, u_2)$, where $u_1 = u_2 = \frac{1}{\sqrt{2}}$. Then $g''_{\boldsymbol{u}}(0) = b$ and $g''_{\boldsymbol{v}}(0) = -b$ have the desired properties.

Suppose that $a \neq 0$. (The case $c \neq 0$ is similar.) Put $\boldsymbol{u} = (1, 0)$ and $\boldsymbol{v} = (v_1, v_2)$, where

$$v_1 = -\frac{b}{\sqrt{a^2+b^2}} \quad \text{and} \quad v_2 = \frac{a}{\sqrt{a^2+b^2}}.$$

Then $g''_{\boldsymbol{u}}(0) = a$ and $g''_{\boldsymbol{v}}(0) = (ac - b^2)\frac{a}{a^2+b^2}$ have the desired properties. ■

The square matrix

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

is called a *Hessian matrix*. Its determinant equals $ac - b^2$.

Example B.6: Find local extrema of the function

$$z = \frac{ax + by + c}{\sqrt{1 + x^2 + y^2}}.$$

If $a = b = c = 0$, the function z is constant, thus it does not attain strict local extrema. Suppose $(a, b, c) \neq (0, 0, 0)$. Then

$$z_1 = \frac{a(1+x^2+y^2) - x(ax+by+c)}{(1+x^2+y^2)^{3/2}}, \quad z_2 = \frac{b(1+x^2+y^2) - y(ax+by+c)}{(1+x^2+y^2)^{3/2}}.$$

Critical points are solutions of the pair of equations

$$z_1 = 0, \quad z_2 = 0,$$

which yields

$$a(1+x^2+y^2) = x(ax+by+c) \text{ and } b(1+x^2+y^2) = y(ax+by+c).$$

Therefore $ay(1+x^2+y^2) = bx(1+x^2+y^2)$, i.e., $ay = bx$. After substituting into the previous equations we obtain

$$cx = a, \qquad cy = b.$$

If $c = 0$, then the function z has no critical points. If $c \neq 0$, then the function z has a critical point $A = (a/c, b/c)$.

Suppose that $c \neq 0$. Then

$$\begin{vmatrix} f_{11}(A) & f_{12}(A) \\ f_{12}(A) & f_{22}(A) \end{vmatrix} = \begin{vmatrix} -\frac{b^2+c^2}{c\left(1+\frac{a^2}{c^2}+\frac{b^2}{c^2}\right)^{3/2}} & \frac{ab}{c\left(1+\frac{a^2}{c^2}+\frac{b^2}{c^2}\right)^{3/2}} \\ \frac{ab}{c\left(1+\frac{a^2}{c^2}+\frac{b^2}{c^2}\right)^{3/2}} & -\frac{a^2+c^2}{c\left(1+\frac{a^2}{c^2}+\frac{b^2}{c^2}\right)^{3/2}} \end{vmatrix} = \frac{a^2+b^2+c^2}{\left(1+\frac{a^2}{c^2}+\frac{b^2}{c^2}\right)^{3/2}} > 0.$$

Evidently, the sign of the second partial derivative

$$f_{11}(A) = -\frac{b^2+c^2}{c\left(1+\frac{a^2}{c^2}+\frac{b^2}{c^2}\right)^{3/2}}$$

is determined by the sign of the parameter c. If $c > 0$, the function z has a strict local maximum at the point A. If $c < 0$, the function z has a strict local minimum at the point A. ◄

Exercises

B.1.1 Find local extrema of the following functions:

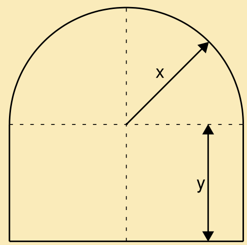

Figure B.1. Window made up of a rectangle and a half-circle

(a) $z = 2x^3 - 3xy + 2y^3 + 1$.

(b) $z = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$.

(c) $z = x^4 + y^4 - 2x^2 + 4xy - 2y^2$.

(d) $z = (y - x^2)(y - 3x^2)$.

(e) $z = (x^2 + y^2)e^{-x^2-y^2}$.

(f) $z = (1 + e^y)\cos x - ye^y$.

(g) $z = x + y + 4\sin x \sin y$.

(h) $z = 81\left(\frac{1}{x} + \frac{1}{y}\right) - (x^2 + xy + y^2)$.

(i) $z = 2 - \sqrt[3]{x^2 + y^2}$.

(j) $z = (x^2 - 1)e^{y^2} - 2x^2$.

B.2 Constrained Extrema

Example B.7: Calculate the dimensions of a window profile according to the picture in Figure B.1 (made up of a rectangle and a half-circle) such that for a given perimeter, it will have the largest possible area.

We wish to maximize the area $A = 2xy + \frac{\pi}{2}x^2$ given that the perimeter $P = 2x + 2y + \pi x$ is constant. After substituting into A for $2y = P - 2x - \pi x$ we have

$$A = Px - 2x^2 - \pi x^2 + \frac{\pi}{2}x^2.$$

◀

By completing the square we obtain

$$A = \frac{P^2}{2(4+\pi)} - \frac{4+\pi}{2}\left(x - \frac{P}{4+\pi}\right)^2.$$

Therefore A attains an absolute maximum

$$A = \frac{P^2}{2(4+\pi)} \text{ at } x = \frac{P}{4+\pi}.$$

Definition B.8: Let $f : D \to \mathbb{R}$, where $D \subset \mathbb{R}^2$. Let $\varphi(x,y) = 0$ be a constraint condition. We say that f has at the point $(x_0, y_0) \in D$ a *constrained maximum*, if $\varphi(x_0, y_0) = 0$ and if $f(x,y) \leq f(x_0, y_0)$ for all $(x,y) \in D$ with $\varphi(x,y) = 0$. If the latter inequality is strict for $(x,y) \neq (x_0, y_0)$, we say that this maximum is a *strict constrained maximum*.

The definition of a *constrained minimum* is similar.

Suppose that the equation $\varphi(x,y) = 0$ can be solved for y as a function of x, i.e., we may consider this level curve as the graph of a function $y = \psi(x)$. Substituting this function into $f(x,y)$ we obtain a function of one variable $f(x, \psi(x))$ and ask for its extrema.

Example B.9: Find the extrema of $z = xy^2$, subject to the constraint $x^2 + y^2 = 9$.

From the constraint equation we have $y^2 = 9 - x^2$. After substituting into $z = xy^2$ for y^2, we obtain $z = 9x - x^3$. Since $z' = 9 - 3x^2$, $z'' = -6x$, the function $z = 9x - x^3$ attains a local maximum at $x = \sqrt{3}$ and a local minimum at $x = -\sqrt{3}$. This yields that the function $z = xy^2$, subject to the constraint $x^2 + y^2 = 9$, attains local maxima at $A = (\sqrt{3}, \sqrt{6})$ and at $B = (\sqrt{3}, -\sqrt{6})$, while it attains local minima at $C = (-\sqrt{3}, \sqrt{6})$ and $D = (-\sqrt{3}, -\sqrt{6})$.

Since $y^2 \geq 0$ for all real numbers y, from the constraint equation we obtain $y^2 = 9 - x^2 \geq 0$, which yields $-3 \leq x \leq 3$. Since for each x, $-3 < x < 0$, we have $z = 9x - x^3 < 0$, the function $z = 9x - x^3$ attains a local maximum $z = 0$ at the endpoint $x = -3$. Since for each x, $0 < x < 3$, we have $z = 9x - x^3 > 0$, the function $z = 9x - x^3$ attains a local minimum at the endpoint $x = 3$. This yields that the function $z = xy^2$, subject to the constraint $x^2 + y^2 = 9$, attains a local maximum at $E = (-3, 0)$ and a local minimum at $F = (3, 0)$. ◀

B.2.1 Lagrange multipliers

Consider level curves of f and the curve C determined by the equation $\varphi(x, y) = 0$. To find maxima of f on the set of all solutions of the equation $\varphi(x, y) = 0$, (i.e., on the curve C) is equivalent to finding a level curve of the greatest level intersecting the curve C.

Theorem B.10: *Let $f : D \to \mathbb{R}$, where $D \subset \mathbb{R}^2$ is an open set. Let (x_0, y_0) is a point of the curve C determined by the equation $\varphi(x, y) = 0$, which lies in the set D. Suppose that the function f restricted to the set C has a local extremum at the point (x_0, y_0). Let f and φ have continuous partial derivatives in some neighborhood of the point (x_0, y_0). Suppose that (x_0, y_0) is not an endpoint of the curve C. Let $\nabla\varphi(x_0, y_0) \neq 0$. Then there is $\lambda_0 \in \mathbb{R}$ such that (x_0, y_0, λ_0) is a critical point of the Lagrange function*

$$L(x, y, \lambda) = f(x, y) + \lambda\varphi(x, y).$$

Proof. Suppose that $\varphi_1(x_0, y_0) \neq 0$. By the Implicit Function Theorem[2] there is an open neighborhood I of the point x_0 and a continuously differentiable function $\psi : I \to \mathbb{R}$ such that $\psi(x_0) = y_0$ and $\varphi(x, \psi(x)) = 0$ for each $x \in I$. Further, by the same theorem for each $x \in I$ we have

$$\psi'(x) = -\frac{\varphi_1(x, \psi(x))}{\varphi_2(x, \psi(x))}.$$

Then by hypothesis the function

$$g(x) = f(x, \psi(x))$$

has a local extremum at the point x_0. This yields that its derivative at the point x_0 equals zero, i.e.,

$$g'(x_0) = f_1(x_0, \psi(x_0)) + f_2(x_0, \psi(x_0))\psi'(x_0) = 0.$$

[2] See Section 12.6 and Exercise 13.11.3.

After substituting $\psi'(x_0)$ we obtain

$$f_1(x_0,\psi(x_0)) - f_2(x_0,\psi(x_0))\frac{\varphi_1(x_0,\psi(x_0))}{\varphi_2(x_0,\psi(x_0))} = 0,$$

that is,

$$f_1(x_0,y_0) - f_2(x_0,y_0)\frac{\varphi_1(x_0,y_0)}{\varphi_2(x_0,y_0)} = 0.$$

Put

$$\lambda_0 = -\frac{f_2(x_0,y_0)}{\varphi_2(x_0,y_0)}.$$

After substituting into the previous inequality we obtain

$$f_1(x_0,y_0) + \lambda_0\varphi_1(x_0,y_0) = 0.$$

After some manipulation, from the definition of λ_0, we obtain

$$f_2(x_0,y_0) + \lambda_0\varphi_2(x_0,y_0) = 0.$$

The proof is complete. ■

B.2.2 Existence of constrained maxima and minima

Now we will deal with a question of existence and type of constrained extrema. We return to the preceding proof. If at the critical point x_0 it is valid that $g''(x_0) < 0$, then the function $g(x) = f(x,\psi(x))$ has at the point x_0 a local maximum.

For each $x \in I$ we have $\varphi(x,\psi(x)) = 0$. Thus the function of one variable

$$h(x) = \varphi(x,\psi(x)), \quad x \in I,$$

is identically zero. Therefore its derivative equals zero, i.e.,

$$h'(x) = \varphi_1(x,\psi(x)) + \varphi_2(x,\psi(x))\psi'(x) = 0.$$

For the function

$$g(x) = f(x,\psi(x)), \quad x \in I,$$

we have

$$g'(x) = f_1(x,\psi(x)) + f_2(x,\psi(x))\psi'(x).$$

We try to express the derivative $g'(x)$ in terms of partial derivatives of the Lagrange function. Compute

$$L_1(x,y,\lambda_0) = f_1(x,y) + \lambda_0\varphi_1(x,y), \quad L_2(x,y,\lambda_0) = f_2(x,y) + \lambda_0\varphi_2(x,y).$$

This yields

$$L_1(x,\psi(x),\lambda_0) = f_1(x,\psi(x)) + \lambda_0\varphi_1(x,\psi(x)),$$
$$L_2(x,\psi(x),\lambda_0) = f_2(x,\psi(x)) + \lambda_0\varphi_2(x,\psi(x)).$$

Then

$$\begin{aligned} &L_1(x,\psi(x),\lambda_0) + L_2(x,\psi(x),\lambda_0)\psi'(x) = \\ &= f_1(x,\psi(x)) + \lambda_0\varphi_1(x,\psi(x)) + \left(f_2(x,\psi(x)) + \lambda_0\varphi_2(x,\psi(x))\right)\psi'(x) = \\ &= \underbrace{f_1(x,\psi(x)) + f_2(x,\psi(x))\psi'(x)}_{=g'(x)} + \lambda_0\Big(\underbrace{\varphi_1(x,\psi(x)) + \varphi_2(x,\psi(x))\psi'(x)}_{=0}\Big). \end{aligned}$$

This shows that

$$g'(x) = L_1(x,\psi(x),\lambda_0) + L_2(x,\psi(x),\lambda_0)\psi'(x).$$

Then for the second derivative of the function g we have

$$\begin{aligned} g''(x) &= L_{11}(x,\psi(x),\lambda_0) + L_{12}(x,\psi(x),\lambda_0)\psi'(x) + \\ &+ \big(L_{12}(x,\psi(x),\lambda_0) + L_{22}(x,\psi(x),\lambda_0)\psi'(x)\big)\,\psi'(x) + L_2(x,\psi(x),\lambda_0)\psi''(x), \end{aligned}$$

thus

$$\begin{aligned} g''(x_0) &= L_{11}(x_0,y_0,\lambda_0) + 2L_{12}(x_0,y_0,\lambda_0)\psi'(x_0) + \\ &+ L_{22}(x_0,y_0,\lambda_0)(\psi'(x_0))^2 + L_2(x_0,y_0,\lambda_0)\psi''(x_0). \end{aligned}$$

Since (x_0,y_0,λ_0) is a critical point of the Lagrange function, we obtain

$$L_2(x_0,y_0,\lambda_0) = 0.$$

Therefore

$$g''(x_0) = L_{11}(x_0,y_0,\lambda_0) + 2L_{12}(x_0,y_0,\lambda_0)\psi'(x_0) + L_{22}(x_0,y_0,\lambda_0)\psi'^2(x_0).$$

Since

$$\psi'(x_0) = -\frac{\varphi_1(x_0,y_0)}{\varphi_2(x_0,y_0)},$$

after some manipulations we obtain

$$g''(x_0) = \frac{1}{\varphi_2^2(x_0,y_0)}\Big[L_{11}(x_0,y_0,\lambda_0)\varphi_2^2(x_0,y_0) -$$

$$-2L_{12}(x_0,y_0,\lambda_0)\varphi_1(x_0,y_0)\varphi_2(x_0,y_0) + L_{22}(x_0,y_0,\lambda_0)\varphi_1^2(x_0,y_0)\Big].$$

If at the stationary point x_0 we have $g''(x_0) < 0$, then the function g has at the point x_0 a local maximum. By the previous formula it is the case if, at the point (x_0, y_0, λ_0), we have

$$L_{11}\varphi_2^2 - 2L_{12}\varphi_1\varphi_2 + L_{22}\varphi_1^2 < 0.$$

This condition may be reformulated by

$$\begin{vmatrix} 0 & \varphi_1 & \varphi_2 \\ \varphi_1 & L_{11} & L_{12} \\ \varphi_2 & L_{12} & L_{22} \end{vmatrix} > 0 \text{ evaluated at the point } (x_0, y_0, \lambda_0).$$

B.2.3 Bordered Hessians

The discussion above can be summarized by the following theorem. The matrix

$$\begin{pmatrix} 0 & \varphi_1 & \varphi_2 \\ \varphi_1 & L_{11} & L_{12} \\ \varphi_2 & L_{12} & L_{22} \end{pmatrix}$$

is called a *bordered Hessian matrix.*

Theorem B.11: *Let $f : D \to \mathbb{R}$, where $D \subset \mathbb{R}^2$ is an open set. Let (x_0, y_0) is a point of the curve C determined by the equation $\varphi(x, y) = 0$, which lies in the set D. Suppose that the function f restricted to the set C has a local extremum at the point (x_0, y_0). Let f and φ have continuous partial derivatives in some neighborhood of the point (x_0, y_0). Suppose that (x_0, y_0) is not an endpoint of the curve C. Let $\nabla\varphi(x_0, y_0) \neq 0$. Let $\lambda_0 \in \mathbb{R}$ be such that (x_0, y_0, λ_0) is a critical point of the Lagrange function*

$$L(x, y, \lambda) = f(x, y) + \lambda\varphi(x, y).$$

Put

$$H^* = \begin{vmatrix} 0 & \varphi_1 & \varphi_2 \\ \varphi_1 & L_{11} & L_{12} \\ \varphi_2 & L_{12} & L_{22} \end{vmatrix} \text{ evaluated at the point } (x_0, y_0, \lambda_0).$$

1. *If $H^* > 0$, then the point (x_0, y_0) is a local maximum of f on C.*
2. *If $H^* < 0$, then the point (x_0, y_0) is a local minimum of f on C.*

Exercises

B.2.1 Find the constrained extrema of the functions on the given curves:

(a) $z = 7x^2 + 4xy + 3y^2$, if $x^2 + y^2 = 1$.

(b) $z = \frac{x-y-4}{\sqrt{2}}$, if $x^2 + y^2 = 1$.

(c) $z = xy$, if $x^2 + y^2 = 1$.

(d) $z = x^2 + 12xy + 2y^2$, if $4x^2 + y^2 = 25$.

(e) $z = \ln(xy)$, if $x^3 + xy + y^3 = 0$.

(f) $z = 1 + \frac{1}{x} + \frac{1}{y}$, if $\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{8}$.

B.2.4 Absolute maxima and minima

Example B.12: We show that

$$\cos\alpha\cos\beta\cos\gamma \leq \frac{1}{8},$$

where α, β, γ are the angles of an arbitrary triangle. Indeed, since $\alpha+\beta+\gamma=\pi$, we have

$$\cos\alpha\cos\beta\cos\gamma = \cos\alpha\cos\beta\cos(\pi-\alpha-\beta) = -\cos\alpha\cos\beta\cos(\alpha+\beta) =$$
$$= -\tfrac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]\cos(\alpha+\beta) = -\tfrac{1}{2}[\cos^2(\alpha+\beta)+\cos(\alpha+\beta)\cos(\alpha-\beta)] =$$

and by completing the square we obtain

$$= -\tfrac{1}{2}[(\cos(\alpha+\beta)+\tfrac{1}{2}\cos(\alpha-\beta))^2 - \tfrac{1}{4}\cos^2(\alpha-\beta)] =$$
$$= \tfrac{1}{8}\cos^2(\alpha-\beta) - \tfrac{1}{2}(\cos(\alpha+\beta)+\tfrac{1}{2}\cos(\alpha-\beta))^2 \le \tfrac{1}{8}\cos^2(\alpha-\beta) \le \tfrac{1}{8}.$$

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Since a continuous function defined on a compact set achieves an absolute maximum (Corollary 11.47), the previous problem can be reformulated as follows:

Find an absolute maximum of the function

$$z = -\cos x\cos y\cos(x+y)$$

subject to the constraints $0 \le x \le \pi$ *and* $0 \le x+y \le \pi$.

Definition B.13: Let f be a function defined on the set $D \subset \mathbb{R}^2$. Let $(x_0, y_0) \in D$. We say that f has an *absolute minimum* on the set D at the point (x_0, y_0), if $f(x_0, y_0) \le f(x, y)$ for each $(x, y) \in D$.

Similarly we define an *absolute maximum.*

Theorem B.14: *Let (x_0, y_0) is an interior point of the set D. If f has an absolute extremum at the point (x_0, y_0), then either $f_1(x_0, y_0) = 0$ and $f_2(x_0, y_0) = 0$ or at least one of those derivatives does not exist.*

Absolute extrema may be located at critical points, points on the boundary of the domain, or points where partial derivatives are not defined.

Exercises

B.2.1 Examine extrema of the function $z = 11 - 8x^2y + 2x^3y + 2x^2y^2$ subject to the constraints $x \ge 0$, $y \ge 0$ and $x + y \le 6$.

B.2.2 Find extrema of the function $z = x^2 + y^2 - x - y + 1$ subject to the constraint $x^2 + y^2 \le 1$.

B.2.3 Find extrema of the function $z = x^2 + y^2 - 12x + 16y + 50$ subject to the constraint $x^2 + y^2 \le 225$.