CONSTRUCTIVE DEFINITIONS FOR NON-ABSOLUTELY CONVERGENT INTEGRALS

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1. Introduction

The usual approaches to generalizations of the classical Denjoy integrals follow the methods of Perron or Ward; typically a convergence factor is introduced to broaden the concept of derivative or increment and an integral descriptively defined by the major–minor function technique. Systematizations of these ideas have been attempted: Jeffery and Miller (**6**) have formalized the notion of convergence factor as it applied to Perron-type integration and Henstock (**4**) has done the same for the Ward integral.

These theories overlook, however, what might be called the geometry of the integration processes; for this reason we have been led in this paper to consider a theory which replaces the usual convergence factor theory. The methods date back to Saks (**10**): a generalization of the classical concept of the oscillation of a function on an interval is presented and integrals are defined constructively using this to obtain Cauchy- and Harnack-type extensions of an integral. The main feature of our theory rests in the emphasis on the topological structure implicit in the definitions; thus, for example, in our context the relation between the Denjoy–Perron and Denjoy–Khintchine integrals depends completely on the concepts of absolute and unconditional convergence in a Banach space of continuous functions.

2. Notation

The symbol $\mathbf{R}$ denotes the set of real numbers; by an *interval* we mean a subset of $\mathbf{R}$ of the form $(a,b) = \{t : a < t < b\}$ where $a, b \in \mathbf{R}$. (In particular the empty set $\emptyset$ is an interval.) If $a \leqslant b \leqslant c$ we agree to write $(a,c) = (a,b)+(b,c)$. The symbol $\mathbf{B}$ denotes an arbitrary, but fixed, real or complex Banach space, $x \to |x|$ the norm function in that space and $\mathbf{B}^*$ the conjugate of $\mathbf{B}$.

A function F whose domain is the family of all subintervals of an interval I and whose range is in $\mathbf{B}$ is said to be an *additive function of intervals on* I if $F(J) = F(J_1)+F(J_2)$ whenever J, J_1, and J_2 are subintervals of I for which $J = J_1+J_2$. In particular $F(\emptyset) = 0$ (the zero element of $\mathbf{B}$) for such a function.

If F is an additive function of intervals on I and I_0 is a subinterval of I then $F(I_0 \cap (:))$ denotes the function $J \to F(I_0 \cap J)$ defined for all intervals J.

We require the following easily proved property of intervals (cf. (**9**)).

(2.1) Let $\mathfrak{F}$ be a family of intervals with the properties: (a) if $J \subseteq J_0 \in \mathfrak{F}$ then $J \in \mathfrak{F}$; (b) if $J = J_1 + J_2$ and J_1, $J_2 \in \mathfrak{F}$ then $J \in \mathfrak{F}$; (c) if $J_0 \in \mathfrak{F}$ whenever $\bar{J}_0 \subseteq J$, then $J \in \mathfrak{F}$. Then $I \subseteq \cup\{J : J \in \mathfrak{F}\}$ implies that $I \in \mathfrak{F}$.

The Lebesgue (Lebesgue–Bochner) integral of a **B**-valued function f defined on a subset E of **R** is denoted by $\mathfrak{L}(f, E)$.

If F is an additive function of intervals on I then the expressions s$DF(t)$, sA$DF(t)$, p$DF(t)$, and pA$DF(t)$ denote respectively the strong derivative, the strong approximate derivative, the pseudo-derivative, and the approximate pseudo-derivative of F in I. These are defined by the following equations:

$$\lim_{h\to 0} |h^{-1}F(I_h) - \mathrm{s}DF(t)| = 0,$$

$$\operatorname*{app\,lim}_{h\to 0} |h^{-1}F(I_h) - \mathrm{sA}DF(t)| = 0,$$

$$\lim_{h\to 0} x^*(h^{-1}F(I_h) - \mathrm{p}DF(t)) = 0 \quad \text{for all } x^* \in B^*,$$

$$\operatorname*{app\,lim}_{h\to 0} x^*(h^{-1}F(I_h) - \mathrm{pA}DF(t)) = 0 \quad \text{for all } x^* \in B^*,$$

when these limits exist for $t \in \bar{I}$, where we have written $I_h = (t, t+h)$ $(h > 0)$ and $I_h = (t+h, t)$ $(h < 0)$.

Finally, we define the (strong) variation of F on an interval I as

$$V(F, I) = \sup\{\sum |F(I \cap J_k)|\},$$

where the supremum is taken with respect to all finite sequences of disjoint intervals $\{J_k\}$.

3. Generalized oscillations

Let two extended-real-valued functions

$$(F, I) \to \omega^r(F, I) \quad \text{and} \quad (F, I) \to \omega^l(F, I)$$

be given, with domains consisting of all pairs (F, I) where I is an arbitrary interval and F is an additive function of intervals on I. If F is an additive function of intervals on I_0, where $I_0 \supseteq I$, and F' denotes the restriction of F to subintervals of I then we shall also write $\omega^s(F', I) = \omega^s(F, I)$. In particular, if F and G agree on subintervals of I, we must have $\omega^s(F, I) = \omega^s(G, I)$. Introduce the auxiliary function $O(F, I) = \max\{\omega^s(F, I) : s = l, r\}$ and denote the pair $\{\omega^r, \omega^l\}$ by **K**. The pair **K** is said to be a *generalized oscillation* if the following five axioms hold.

I. For all intervals I, all additive functions of intervals F and G on I, all scalars c, and $s = l, r$,

(i) $0 \leqslant \omega^s(F, I) \leqslant +\infty$,

(ii) $\omega^s(cF, I) = |c|\,\omega^s(F, I)$ $[0.\infty = 0]$,

(iii) $\omega^s(F+G, I) \leqslant \omega^s(F, I) + \omega^s(G, I)$.

II. The expressions $\omega^r(F, (x, x+h))$ and $\omega^l(F, (x-h, x))$ are monotone non-decreasing functions of h ($\geqslant 0$) for each x and F for which they are defined.

III. If $F(a, a+h)$ is a non-zero constant for all sufficiently small $h > 0$, then $\limsup_{h \to 0+} \omega^r(F, (a, a+h)) > 0$. Similarly, if $F(b-h, b)$ is a non-zero constant for all sufficiently small $h > 0$, then $\limsup_{h \to 0+} \omega^l(F, (b-h, b)) > 0$.

IV. There is a constant C independent of F and I such that

$$|F(I)| \leqslant CO(F, I).$$

V. If $\{I_k\}$ is a finite or infinite sequence of disjoint intervals and the series $\sum F(I_k \cap J)$ converges unconditionally for every subinterval J of I then

$$O(\sum F(I_k \cap (\,:\,)), I) \leqslant C \sum O(F, I_k \cap I),$$

where C is a constant independent of F, I and the sequence $\{I_k\}$. Note in particular that $\omega^r(F, \emptyset) = \omega^l(F, \emptyset) = 0$ as a consequence of I(ii).

Definition 1. $\mathfrak{B}_I(\mathbf{K})$ denotes the topological vector space of all additive functions of intervals F on an interval I for which $O(F, J) < +\infty$ for each $J \subseteq I$, with the topology generated by the collection of semi-norms $\{F \to O(F, J) : J \subseteq I\}$. More precisely a base for the system of neighbourhoods of the origin is given by all finite intersections of sets of the form $\{F : O(F, J) < \varepsilon\}$ for arbitrary $\varepsilon > 0$ and $J \subseteq I$.

Definition 2. A function F in $\mathfrak{B}_I(\mathbf{K})$ is said to be **K**-*continuous on* I if, for every $t \in \mathbf{R}$,

$$\lim_{h \to 0+} \omega^r(F, I \cap (t+h)) = \lim_{h \to 0+} \omega^l(F, I \cap (t-h, t)) = 0.$$

The class of all functions **K**-continuous on I is clearly a linear subspace of $\mathfrak{B}_I(\mathbf{K})$ and, with the relative topology, is denoted by $\mathfrak{C}_I(\mathbf{K})$.

The following lemmas may be proved directly from the axioms.

LEMMA 3.1. *$\mathfrak{B}_I(\mathbf{K})$ is a locally convex separated space and $\mathfrak{C}_I(\mathbf{K})$ is a closed linear subspace of $\mathfrak{B}_I(\mathbf{K})$.*

Proof. Certainly $\mathfrak{B}_I(\mathbf{K})$ is locally convex and axiom IV shows that it is separated. To see that $\mathfrak{C}_I(\mathbf{K})$ is closed let $\{F_n\}$ be a (generalized) sequence of elements in $\mathfrak{C}_I(\mathbf{K})$ converging to a function F in $\mathfrak{B}_I(\mathbf{K})$. Then $\lim O(F-F_n, J) = 0$ for each $J \subseteq I$. In particular, for arbitrary $a \in \bar{I}$ and $\varepsilon > 0$, there exists N such that $\omega^r(F-F_N, (a, a+h)) < \varepsilon/2$ (cf. axiom III) for all $(a, a+h) \subseteq I$.

Choose $\delta > 0$ such that $\omega^r(F_N, (a, a+h)) < \varepsilon/2$ whenever $0 < h < \delta$, $(a, a+h) \subseteq I$; then certainly (by I(iii)) we have $\omega^r(F, (a, a+h)) < \varepsilon$ whenever $0 < h < \delta$, $(a, a+h) \subseteq I$. Similar arguments apply at each point in $\bar{I}$ and on either side so that F is $\mathbf{K}$-continuous on I and hence is in $\mathfrak{C}_I(\mathbf{K})$ as required.

LEMMA 3.2. *If $J \subseteq I$ and F is in $\mathfrak{B}_J(\mathbf{K})$ (resp. $\mathfrak{C}_J(\mathbf{K})$) then $F(J \cap (:))$ is in $\mathfrak{B}_I(\mathbf{K})$ (resp. $\mathfrak{C}_I(\mathbf{K})$).*

We require now a general concept of an integral; the definitions which follow are natural extensions of those in Saks (**10**). Let $\mathfrak{J}$ be a $\mathbf{B}$-valued function whose domain, $\operatorname{dom}\mathfrak{J}$, is a set of ordered pairs $\{(f, I)\}$ where f is a $\mathbf{B}$-valued function defined on an interval I. For convenience we write $\operatorname{dom}_I \mathfrak{J} = \{f\colon (f, I) \in \operatorname{dom}\mathfrak{J}\}$. The operation $\mathfrak{J}$ is called a $\mathbf{K}$-integral if each of the following conditions holds.

(i) If $(f, I) \in \operatorname{dom}\mathfrak{J}$ then $(f, J) \in \operatorname{dom}\mathfrak{J}$ for every $J \subseteq I$ and

$$\mathfrak{J}(f, I \cap (:)) \in \mathfrak{C}_I(\mathbf{K}).$$

(ii) If $I = I_1 + I_2$ and $(f, I_i) \in \operatorname{dom}\mathfrak{J}$ $(i = 1, 2)$ then $(f, I) \in \operatorname{dom}\mathfrak{J}$.

(iii) If $f = 0$ on I then $(f, I) \in \operatorname{dom}\mathfrak{J}$ and $\mathfrak{J}(f, I) = 0$.

The standard terminology of integration theory is used freely: thus a function f in $\operatorname{dom}_I \mathfrak{J}$ is said to be $\mathfrak{J}$-integrable on I and a point $a \in \bar{I}$ is called an $\mathfrak{J}$-singular point of f in I if there exist arbitrarily small intervals $J \subseteq I$ with $a \in \bar{J}$ such that $f \notin \operatorname{dom}_J \mathfrak{J}$.

Now suppose that some well-defined $\mathbf{K}$-integral $\mathfrak{J}$ is given; the definitions which follow describe various extensions of $\mathfrak{J}$ which are possible in our context.

DEFINITION 3. $\operatorname{dom}_I \mathfrak{J}^{C(\mathbf{K})}$ is defined to be the class of all functions f on I for which

(i) the set S of all $\mathfrak{J}$-singular points of f in I is finite, and

(ii) there is a function $F \in \mathfrak{C}_I(\mathbf{K})$ such that $F(J) = \mathfrak{J}(f, J)$ for every $J \subseteq I$ for which $\bar{J} \cap S = \emptyset$.

When this is the case we define

(iii) $\mathfrak{J}^{C(\mathbf{K})}(f, I) = F(I)$.

DEFINITION 4. We define $\mathrm{dom}_I \mathfrak{J}^{H(\mathbf{K})}$ to be the class of all functions f on I for which

(i) if S denotes the (necessarily closed) set of all $\mathfrak{J}$-singular points of f in I and χ_S the characteristic function of S then $f\chi_S$ is $\mathfrak{J}$-integrable in I, and

(ii) if $\{I_k\}$ is the sequence of intervals complementary to S in I then $(f, I_k) \in \mathrm{dom}\,\mathfrak{J}$ for each k and the series $\sum \mathfrak{J}(f, I_k \cap (:))$ converges unconditionally in $\mathfrak{B}_I(\mathbf{K})$.

When this is the case we define

(iii) $\mathfrak{J}^{H(\mathbf{K})}(f, I) = \mathfrak{J}(f\chi_S, I) + \sum \mathfrak{J}(f, I_k)$.

If (ii) above is strengthened by requiring also that $\sum |\mathfrak{J}(f, I_k)| < +\infty$ then the resulting extension will be denoted by $\mathfrak{J}^{H_s(\mathbf{K})}$; finally, if (ii) is replaced by the stronger requirement that $\sum O(\mathfrak{J}(f, (:)), I_k) < +\infty$ then the resulting extension will be denoted by $\mathfrak{J}^{H_*(\mathbf{K})}$. That this requirement is in fact stronger is proved in (3.7).

THEOREM 3.3. *If $\mathfrak{J}$ is a well-defined* **K***-integral then so also is each of the extensions $\mathfrak{J}^{C(\mathbf{K})}$, $\mathfrak{J}^{H(\mathbf{K})}$, $\mathfrak{J}^{H_s(\mathbf{K})}$, and $\mathfrak{J}^{H_*(\mathbf{K})}$.*

Proof. If the function F which appears in Definition 3 is unique then it is easily verified that $\mathfrak{J}^{C(\mathbf{K})}$ is a **K**-integral and that in fact $\mathfrak{J}^{C(\mathbf{K})}(f, J) = F(J)$ for all $J \subseteq I$ and fixed $f \in \mathrm{dom}_I \mathfrak{J}^{C(\mathbf{K})}$. That F is unique is a result of the following assertion, for the general case may be reduced to the case in which S, the set of $\mathfrak{J}$-singular points of f in I, consists of a single point at either end of the interval I.

(3.4) *Let F and G be in $\mathfrak{C}_I(\mathbf{K})$ and suppose that $F(J) = G(J)$ whenever $\bar{J} \subseteq I$; then $F = G$ on I.*

Let $H = F - G$ and write $I = (a, b)$; we see that H is **K**-continuous on I and that, if $0 < h < k < b-a$,

$$H(a, a+k) = H(a, a+h) + H(a+h, a+k) = H(a, a+h).$$

Hence $H(a, a+k)$ is independent of k $(0 < k < b-a)$ and accordingly, by axiom III, H cannot be **K**-continuous on I unless $H(a, a+k) = 0$ whenever $0 < k < b-a$. Similar arguments apply at the right-hand end-point of I, so that H vanishes on I, which proves (3.4).

To establish that the integral $\mathfrak{J}^{H(\mathbf{K})}$ of Definition 4 is a well-defined **K**-integral we first observe the following facts (using the notation of Definition 4).

(3.5) *If the series $\sum \mathfrak{J}(f, I_k \cap (:))$ converges unconditionally in $\mathfrak{B}_I(\mathbf{K})$ to a function F, then $F \in \mathfrak{C}_I(\mathbf{K})$ and the series $\sum \mathfrak{J}(f, I_k \cap J)$ converges unconditionally in* **B** *to $F(J)$ for each $J \subseteq I$.*

(3.6) *For each* $J \subseteq I$,

$$f \in \operatorname{dom}_J \mathfrak{J}^{H(\mathbf{K})} \quad \textit{and} \quad \mathfrak{J}^{H(\mathbf{K})}(f, J) = \mathfrak{J}(f\chi_S, J) + \sum \mathfrak{J}(f, I_k \cap J).$$

Assertion (3.5) is a direct consequence of axiom IV, the fact that each $\mathfrak{J}(f, I_k \cap (:)) \in \mathfrak{C}_I(\mathbf{K})$ and that this latter space is closed in $\mathfrak{B}_I(\mathbf{K})$; the second assertion is a direct consequence of the definition of $\mathfrak{J}^{H(\mathbf{K})}$. From the above it is easy to see that $\mathfrak{J}^{H(\mathbf{K})}$ is a **K**-integral; a similar result holds for the integral $\mathfrak{J}^{H_*(\mathbf{K})}$.

Finally, for the integral $\mathfrak{J}^{H^*(\mathbf{K})}$, the proof follows as above with the aid of the following fact.

(3.7) *If* $\{I_k\}$ *is a sequence of disjoint intervals, if* $F(I_k \cap (:)) \in \mathfrak{C}_I(\mathbf{K})$ *for each* k *and if* $\sum O(F, I_k \cap I) < +\infty$, *then the series* $\sum F(I_k \cap (:))$ *converges unconditionally in* $\mathfrak{C}_I(\mathbf{K})$ *and* $\sum |F(I_k \cap I)| < +\infty$.

By axiom IV, $\sum |F(I_k \cap I)| \leqslant C \sum O(F, I_k \cap I) < +\infty$ so that a function G may be defined on subintervals of I by writing $G(J) = \sum F(I_k \cap J)$ for each $J \subseteq I$ since this series converges absolutely for each such J. Now write $F_n(:) = \sum_{k \leqslant n} F(I_k \cap (:))$ and use axiom V to obtain the inequality

$$O(F - F_n, J) \leqslant C \sum_{k>n} O(F, I_k \cap J) \quad (J \subseteq I).$$

The right-hand side of this inequality tends to zero with increasing n which verifies that $F_n \to G$ in $\mathfrak{C}_I(\mathbf{K})$; as this argument applies to any rearrangement of the indices k we have proved (3.7).

A generalized totalization process is now defined by applying the methods of Saks (**10**) to the above extension procedures. Let $\mathfrak{J}_1$ and $\mathfrak{J}_2$ be **K**-integrals: we shall write $\mathfrak{J}_1 \subseteq \mathfrak{J}_2$ and say that $\mathfrak{J}_2$ includes $\mathfrak{J}_1$ provided that whenever $(f, I) \in \operatorname{dom} \mathfrak{J}_1$ then necessarily $(f, I) \in \operatorname{dom} \mathfrak{J}_2$ and $\mathfrak{J}_1(f, I) = \mathfrak{J}_2(f, I)$. Suppose that $\{\mathfrak{J}^\alpha\}$ is a transfinite sequence of **K**-integrals such that $\mathfrak{J}^\alpha \subseteq \mathfrak{J}^\beta$ whenever $\alpha < \beta$. We define the integral $\sum_{\alpha<\beta} \mathfrak{J}^\alpha$ by requiring that

(i) $$\operatorname{dom} \sum_{\alpha<\beta} \mathfrak{J}^\alpha = \bigcup_{\alpha<\beta} \operatorname{dom} \mathfrak{J}^\alpha$$

and

(ii) $$\sum_{\alpha<\beta} \mathfrak{J}^\alpha(f, I) = \mathfrak{J}^{\alpha_0}(f, I)$$

where $\alpha_0 < \beta$ is any ordinal for which $(f, I) \in \operatorname{dom} \mathfrak{J}^{\alpha_0}$.

DEFINITION 5. Suppose that $\mathfrak{J}$ is a **K**-integral. Then the following extensions of $\mathfrak{J}$ are defined:

(i) the **K**-total of $\mathfrak{J}$ is the integral $\sum_{\alpha<\Omega} \mathfrak{J}^\alpha$,

(ii) the $\mathbf{K}_s$-total of $\mathfrak{I}$ is the integral $\sum\limits_{\alpha<\Omega} \mathfrak{I}_s^{\alpha}$,

(iii) the $\mathbf{K}_*$-total of $\mathfrak{I}$ is the integral $\sum\limits_{\alpha<\Omega} \mathfrak{I}_*^{\alpha}$,

where Ω denotes the first uncountable ordinal and where we write $\mathfrak{I}^0 = \mathfrak{I}_s^{\,0} = \mathfrak{I}_*^{\,0} = \mathfrak{I}$ and, inductively,

$$\mathfrak{I}^{\beta} = \left(\left[\sum_{\alpha<\beta} \mathfrak{I}^{\alpha}\right]^{C(\mathbf{K})}\right)^{H(\mathbf{K})},$$

$$\mathfrak{I}_s^{\beta} = \left(\left[\sum_{\alpha<\beta} \mathfrak{I}_s^{\alpha}\right]^{C(\mathbf{K})}\right)^{H_s(\mathbf{K})},$$

$$\mathfrak{I}_*^{\beta} = \left(\left[\sum_{\alpha<\beta} \mathfrak{I}_*^{\alpha}\right]^{C(\mathbf{K})}\right)^{H_*(\mathbf{K})}.$$

In the next section these processes will be applied to extend the Lebesgue integral: one general result, however, is easily proved now (cf. ((**1**) 121)).

THEOREM 3.8. *Let $\mathfrak{I}$ be a* $\mathbf{K}$*-integral which has the property that every function which is $\mathfrak{I}$-integrable on an interval is strongly measurable on that interval. Then the* $\mathbf{K}$-, $\mathbf{K}_s$- *and* $\mathbf{K}_*$*-totals of $\mathfrak{I}$ have the same property.*

4. The Denjoy integrals

Suppose that a generalized oscillation $\mathbf{K}$ has been defined which in addition to satisfying axioms I–V satisfies also the following axiom.

VI. There is a constant C such that, for every interval I and for every additive function of intervals F on I,

$$O(F,I) \leqslant CV(F,I).$$

This assumption guarantees that the Lebesgue integral, which we have denoted by $\mathfrak{L}$, is a $\mathbf{K}$-integral in the sense of the previous section; this permits our definitions.

DEFINITION 6. The integrals $\mathfrak{D}(\mathbf{K})$, $\mathfrak{D}_s(\mathbf{K})$, and $\mathfrak{D}_*(\mathbf{K})$ are the $\mathbf{K}$-, $\mathbf{K}_s$-, and $\mathbf{K}_*$-totals of the integral $\mathfrak{L}$.

THEOREM 4.1. *If a function is $\mathfrak{D}(\mathbf{K})$, $\mathfrak{D}_s(\mathbf{K})$, or $\mathfrak{D}_*(\mathbf{K})$-integrable on an interval then it is necessarily strongly measurable on that interval.*

Proof. Certainly the $\mathfrak{L}$ integral has this property and hence this result is contained in Theorem 3.8.

THEOREM 4.2. $\mathfrak{D}_*(\mathbf{K}) \subseteq \mathfrak{D}_s(\mathbf{K}) \subseteq \mathfrak{D}(\mathbf{K})$.

Proof. In fact it is apparent from the definitions of these totals that the same relation exists in general between the $\mathbf{K}_*$-, $\mathbf{K}_s$-, and $\mathbf{K}$-totals of any $\mathbf{K}$-integral.

DEFINITION 7. A function F in $\mathfrak{C}_I(\mathbf{K})$ is said to be a *$\mathbf{K}$-primitive of a function f on I* if, for every closed set E in $\mathbf{R}$ which contains points of I, there is an interval $J \subseteq I$ containing points of E for which

(i) $f\chi_E$ is $\mathfrak{L}$-integrable on J,

(ii) the series $\sum F(J_k \cap (:))$ converges unconditionally in $\mathfrak{C}_J(\mathbf{K})$ where $\{J_k\}$ are the intervals complementary to E in J, and

(iii) for every $J' \subseteq J$,

$$F(J') = \mathfrak{L}(f\chi_E, J') + \sum F(J_k \cap J').$$

If (ii) above is strengthened by requiring also that $\sum |F(J_k)| < +\infty$ then F is said to be a *$\mathbf{K}_s$-primitive* of f on I; finally if (ii) is replaced by the stronger condition (cf. (3.7)) that $\sum O(F, J_k) < +\infty$ then F is said to be a *$\mathbf{K}_*$-primitive* of f on I.

The principal result of this section is a descriptive characterization of the previously defined integrals using the concepts of Definition 7; the formal ideas date back to Romanovski (8).

THEOREM 4.3. *A function f is $\mathfrak{D}(\mathbf{K})$-integrable (resp. $\mathfrak{D}_s(\mathbf{K})$-, $\mathfrak{D}_*(\mathbf{K})$-integrable) on an interval I if and only if there exists a function F which is a $\mathbf{K}$-primitive (resp. $\mathbf{K}_s$-, $\mathbf{K}_*$-primitive) of f on I. Then F is the indefinite integral of f on I.*

Proof. We firstly establish the fact that the function F appearing in the statement of the theorem is necessarily unique; this then permits an integral to be defined using the concept of a $\mathbf{K}$-primitive. The content of our proof is simply that this integral is equivalent to $\mathfrak{D}(\mathbf{K})$.

(4.4) *Let F and G be $\mathbf{K}$-primitives for f on I. Then $F = G$ on I.*

Let $H = F - G$. Then it may be shown that H is a $\mathbf{K}$-primitive of $g = 0$ on I; we need only show that H vanishes on I. To this end let $\mathfrak{F}$ denote the family of subintervals of I on which H vanishes identically. This family evidently has the properties of (2.1) (a), (b), and (c): (a) and (b) are trivial while (c) has been proved in (3.4). Let E be the complement of the set $\bigcup \{J : J \in \mathfrak{F}\}$; E is closed, and if E contains points of I then, since H is a $\mathbf{K}$-primitive of $g = 0$ in I, there is an interval $J \subseteq I$ containing points of E for which

$$H(J') = L(g\chi_E, J') + \sum H(J_k \cap J') \quad (J' \subseteq J).$$

But $g = 0$ and each interval J_k in $J \backslash E$ belongs to $\mathfrak{F}$, so that the entire right-hand side vanishes for each $J' \subseteq J$. Hence, by definition, $J \in \mathfrak{F}$

which contradicts the fact that $E \cap J \neq \emptyset$. Thus E does not contain points of I and hence, by (2.1), $I \in \mathfrak{F}$ which proves (4.4).

Let an integral $\mathfrak{J}$ be defined: a function f will be $\mathfrak{J}$-integrable on an interval I if and only if there exists a function F which is a $\mathbf{K}$-primitive of f on I. By (4.4) such a function, if it exists, is unique and we put $\mathfrak{J}(f, I) = F(I)$; a direct verification will show that $\mathfrak{J}$ is even a $\mathbf{K}$-integral. The theorem is obviously proved if we show that $\mathfrak{J} = \mathfrak{D}(\mathbf{K})$.

$$(4.5) \qquad \mathfrak{J} \supseteq \mathfrak{L}.$$

This follows from axiom VI and properties of the $\mathfrak{L}$-integral.

$$(4.6) \qquad \mathfrak{J}^{C(\mathbf{K})} = \mathfrak{J}.$$

Suppose that $f \in \operatorname{dom}_I \mathfrak{J}^{C(\mathbf{K})}$, $F(\,:\,) = \mathfrak{J}^{C(\mathbf{K})}(f, (\,:\,))$ and that S is the (finite) set of $\mathfrak{J}$-singular points of f in I. Every perfect set E (there is no loss of generality in Definition 7 in ignoring isolated points) has a portion $E \cap J$ with $J \subseteq I$ and $\bar{J} \cap S = \emptyset$. But F is a $\mathbf{K}$-primitive of f on every such interval J so that by taking a further portion if necessary we can demand that F satisfy conditions (i)–(iii) of Definition 7 with respect to f and E on J. Thus F is a $\mathbf{K}$-primitive of f on I and $f \in \operatorname{dom}_I \mathfrak{J}$ which proves (4.6).

$$(4.7) \qquad \mathfrak{J}^{H(\mathbf{K})} = \mathfrak{J}.$$

Suppose that $f\chi_S$ is $\mathfrak{J}$-integrable on I, that f is $\mathfrak{J}$-integrable on each interval I_k in $I \backslash S$ where S is a closed set, and that the series $\sum \mathfrak{J}(f, I_k \cap (\,:\,))$ converges unconditionally in $\mathfrak{C}_I(\mathbf{K})$. We write

$$F(J) = \mathfrak{J}(f\chi_S, J) + \sum \mathfrak{J}(f, I_k \cap J) \quad (J \subseteq I)$$

and prove that in this case F is necessarily a $\mathbf{K}$-primitive of f on I which proves that $f \in \operatorname{dom}_I \mathfrak{J}$; this will establish (4.7). Without loss of generality we may take $f = 0$ on S.

Let E be a perfect set containing points of I; by Baire's theorem some portion $E \cap J$ $(J \subseteq I)$ is contained entirely within one of the sets S, I_1, I_2, ...; if $E \cap J \subseteq I_k$ then the arguments of (4.6) will show that F satisfies conditions (i)–(iii) of Definition 7 with respect to f and E on J. If $E \cap J \subseteq S$ then, denoting the intervals in $J \backslash E$ by $\{J_m\}$, we verify that $f\chi_E$ is $\mathfrak{L}$-integrable on J (trivially) and that the series $\sum F(J_m \cap (\,:\,))$ converges unconditionally in $\mathfrak{C}_I(\mathbf{K})$. This latter fact is proved by the relation

$$\sum_{m \in \pi} F(J_m \cap (\,:\,)) = \sum \{F(I_k \cap (\,:\,)) : I_k \subseteq J_m,\ m \in \pi\}$$

where π is any finite set of indices m (note that each J_m has end-points in S) and by the fact that $\sum F(I_k \cap (\,:\,))$ converges unconditionally in

$\mathfrak{C}_I(\mathbf{K})$ by hypothesis. Thus for every $J' \subseteq J$ we have

$$F(J') = \mathfrak{L}(f\chi_E, J') + \sum F(J_m \cap J').$$

This verifies conditions (i)–(iii) of Definition 7 in either case and hence F is a $\mathbf{K}$-primitive of f in I as required.

(4.8) $$\mathfrak{D}(\mathbf{K}) \subseteq \mathfrak{I}.$$

Transfinite induction using (4.5), (4.6), and (4.7) proves this assertion.

(4.9) $$\mathfrak{D}(\mathbf{K}) = \mathfrak{I}.$$

The arguments of ((**10**) 258–59) prove this, completing the proof of the theorem for the $\mathfrak{D}(\mathbf{K})$-integral; essentially the same steps with the obvious modifications may be carried out for the other integrals.

5. Differential properties

In this section we present characterizations of the $\mathfrak{D}_s(\mathbf{K})$- and $\mathfrak{D}_*(\mathbf{K})$-integrals in the same spirit as the classical descriptive theory for the Denjoy integrals. The definitions and notations follow the approach used in (**11**) to obtain similar results for the Cesàro–Perron integrals. Several elementary results on the differentiation of interval functions with values in a Banach space are taken from (**1**): for real-valued functions these simplify to well-known classical results and should cause no difficulty to the reader who is interested solely in this simpler situation.

An additive function of intervals F on an interval I is said to be *sAC* (resp. *AC*) *on a set E in I* if, for every $\varepsilon > 0$, there is a $\delta > 0$ such that $\sum |F(J_k \cap I)| < \varepsilon$ (resp. $|\sum F(J_k \cap I)| < \varepsilon$) for every finite sequence of disjoint intervals $\{J_k\}$ with end-points in E for which $\sum m(J_k) < \delta$ (m here denotes Lebesgue measure); if F is sAC on $\mathbf{R}$ in I then we simply say that F is sAC in I.

Definition 8. An additive function of intervals F on I is said to be *AC*($\mathbf{K}$-*sense*) (resp. AC_s($\mathbf{K}$-*sense*); AC_*($\mathbf{K}$-*sense*)) *on a set E in I* if, for every $\varepsilon > 0$ and every interval $I' \subseteq I$, there is a $\delta > 0$ such that

(i) $O(\sum F(J_k \cap (:)), I') < \varepsilon$

(resp. (i)$_s$ $O(\sum F(J_k \cap (:)), I') < \varepsilon$ and $\sum |F(J_k \cap I)| < \varepsilon$;

(i)$_*$ $\sum O(F, J_k \cap I) < \varepsilon$)

for every finite sequence of disjoint intervals $\{J_k\}$ with end-points in E for which $\sum m(J_k) < \delta$.

An additive function of intervals F on I is said to be [*sACG*] (resp. [*ACG*]) in I if there is a sequence of closed sets $\{E_n\}$ with $\bigcup E_n = \mathbf{R}$ such that F is sAC (resp. AC) on each set E_n in I.

DEFINITION 9. A function F in $\mathfrak{C}_I(\mathbf{K})$ is said to be *ACG*(**K**-*sense*) (resp. ACG_s(**K**-*sense*); ACG_*(**K**-*sense*)) *in I* if there is a sequence of closed sets $\{E_n\}$ with $\bigcup E_n = \mathbf{R}$ such that F is AC(**K**-sense) (resp. AC_s(**K**-sense); AC_*(**K**-sense)) on each set E_n in I.

Following (**10**) Theorem 9.1, p. 233 we may prove

LEMMA 5.1. *A function F in $\mathfrak{C}_I(\mathbf{K})$ is ACG(**K**-sense) in I if and only if for every closed set E containing points of I there is an interval $J \subseteq I$ containing points of E such that F is AC(**K**-sense) on E in J. The corresponding statements are true for the concepts ACG_s(**K**-sense) and ACG_*(**K**-sense).*

LEMMA 5.2. *A function which is ACG(**K**-sense) in an interval is necessarily [ACG] in that interval. A function which is ACG_s(**K**-sense) or ACG_*(**K**-sense) in an interval is necessarily* [s*ACG*] *in that interval.*

Proof. This is an easy consequence of axiom IV.

The fundamental result of this section which we now state and prove depends largely on the well-known descriptive characterization of the Lebesgue–Bochner integral (cf. (**5**)).

(5.3) *A function f is $\mathfrak{L}$-integrable on an interval I if and only if there exists an additive function of intervals F which is sAC in I and for which* $\mathrm{s}DF(t) = f(t)$ *a.e. in I. Then necessarily $F(J) = \mathfrak{L}(f, J)$ for each interval $J \subseteq I$.*

THEOREM 5.4. *A function f is $\mathfrak{D}_s(\mathbf{K})$-integrable (resp. $\mathfrak{D}_*(\mathbf{K})$-integrable) on an interval I if and only if there exists a function F that is ACG_s(**K**-sense) (resp. ACG_*(**K**-sense)) in I such that* $\mathrm{s}ADF(t) = f(t)$ *a.e. in I. Then necessarily $F(J) = \mathfrak{D}_s(\mathbf{K})(f, J)$ for each interval $J \subseteq I$.*

Proof. We prove the theorem only for the $\mathfrak{D}_*(\mathbf{K})$-integral; the other case is similar and is omitted. Suppose firstly that $(f, I) \in \operatorname{dom} \mathfrak{D}_*(\mathbf{K})$. Then, by Theorem 4.3, for every closed set E containing points of I there is an interval $J \subseteq I$ containing points of E such that (i) $f\chi_E$ is $\mathfrak{L}$-integrable on J, (ii) $\sum O(F, J_k) < +\infty$ (where $\{J_k\}$ are the intervals in $J \backslash E$ and $F = \mathfrak{D}_*(\mathbf{K})(f, (\,:\,)))$, and (iii) $F(J') = \mathfrak{L}(f\chi_E, J') + \sum F(J_k \cap J')$ for each $J' \subseteq J$.

Define $G_1(J') = \mathfrak{L}(f\chi_E, J')$ and $G_2(J') = \sum F(J_k \cap J')$ $(J' \subseteq J)$; the function G_1 is clearly AC_*(**K**-sense) on E in J (as a result of axiom VI) and we proceed to show that the same is true for G_2. Given any $\varepsilon > 0$ there is, by (ii) above, an integer N so that $\sum_{k \geqslant N} O(F, J_k) < \varepsilon/C$ (where C is the constant of axiom V). Choose $\delta > 0$ so that $\delta < \min\{m(J_k);\, k < N\}$;

then if $\{I_j\}$ is a finite sequence of disjoint intervals with end-points in E and such that $\sum m(I_j) < \delta$ we have, by axiom V,

$$\sum O(G_2, I_j \cap J) \leqslant C \sum \{O(F, J_k) \colon J_k \subseteq I_j \text{ for some } j\}$$

and by the choice of δ the right-hand side of this expression is smaller than ε; hence G_2 is AC_*(**K**-sense) on E in J. Since $F = G_1 + G_2$ it can be shown that F is AC_*(**K**-sense) on E in J: by Lemma 5.1 then it follows that F is ACG_*(**K**-sense) in I.

We continue by defining a function g by putting $g(t) = f(t)$ $(t \in E \cap J)$ and $g(t) = F(J_k)/m(J_k)$ $(t \in J_k)$; since f and hence g are $\mathfrak{L}$-integrable on $E \cap J$ and since $\sum |F(J_k)| < +\infty$ it follows that g is $\mathfrak{L}$-integrable on J and moreover that $\mathrm{s}DG(t) = g(t)$ a.e. in J where $G = \mathfrak{L}(g, (:))$. But because of property (iii) above F and G coincide on intervals with end-points in $\bar{J} \cap E$ and hence $\mathrm{s}ADF(t) = \mathrm{s}D_E F(t) = \mathrm{s}DG(t) = f(t)$ a.e. in $J \cap E$.

In general, then, every closed set E contains a portion $J \cap E$ on which $\mathrm{s}ADF(t) = f(t)$ a.e.; standard arguments may be employed to show that this equality holds a.e. in J. This establishes the necessity of the assertion of the theorem for the D_*(**K**)-integral.

Conversely, suppose that F is ACG_*(**K**-sense) in I and that $\mathrm{s}ADF(t) = f(t)$ a.e. in I: then for every closed set E containing points of I there is, by Lemma 5.1, an interval $J \subseteq I$ containing points of E such that F is AC_*(**K**-sense) on E in J. In particular it can be shown that F is sAC on E and J and that $\sum O(F, J_k) < +\infty$ where $\{J_k\}$ are the intervals in $J \backslash E$.

Define an additive function of intervals G on J by requiring that F and G coincide on $J \cap E$ and that G be linear on the intervals J_k: more precisely, if $J = (\alpha, \beta)$ and $J_k = (\alpha_k, \beta_k)$, we define $G(\alpha, t) = F(\alpha, t)$ $(t \in \bar{J} \cap E)$ and $G(\alpha, t) = F(\alpha, \alpha_k) + F(J_k)[(t - \alpha_k)/(\beta_k - \alpha_k)]$ $(t \in J_k)$, and require that G be additive on J.

The function G is certainly sAC on E in J and, since $\sum |F(J_k)| < +\infty$, elementary arguments show that G is even sAC in J (cf. (**1**) (2.1), p. 101). Further, $\mathrm{s}ADF(t)$ exists by hypothesis a.e. in J so that by the construction of G we see that $\mathrm{s}ADG(t)$ also exists a.e. in J; since G is sAC in J this implies that $\mathrm{s}DG(t)$ exists a.e. in J ((**1**) Theorem 9, p. 107).

Now, writing $g(t) = \mathrm{s}DG(t)$ a.e. in J, we may show directly that, a.e. in J, $g(t) = f(t)$ $(t \in E)$ and $g(t) = F(J_k)/m(J_k)$ $(t \in J_k)$. By (5.3) then g is $\mathfrak{L}$-integrable in J and in fact, for every interval $J' \subseteq J$ with endpoints in E, $G(J') = \mathfrak{L}(f\chi_E, J') + \sum F(J_k \cap J')$. A direct computation then shows that for every interval $J' \subseteq J$, $F(J') = \mathfrak{L}(f\chi_E, J') + \sum F(J_k \cap J')$. Since E was an arbitrary closed set the preceding has established that F is a $\mathbf{K}_*$-primitive of f in I (Definition 7) and hence, by Theorem 4.3, that

f is $\mathfrak{D}_*(\mathbf{K})$-integrable in I and $F = \mathfrak{D}_*(\mathbf{K})(f, (:))$. This completes the proof of the theorem.

A similar result is not available in general for the $\mathfrak{D}(\mathbf{K})$-integral: of course if the underlying Banach space $\mathbf{B}$ is finite-dimensional then the $\mathfrak{D}_s(\mathbf{K})$- and $\mathfrak{D}(\mathbf{K})$-integrals coincide. The theorem below asserts a necessary but not sufficient condition.

THEOREM 5.5. *Let* $(f, I) \in \operatorname{dom} \mathfrak{D}(\mathbf{K})$ *and suppose that* $F(J) = \mathfrak{D}(\mathbf{K})(f, J)$ *for all* $J \subseteq I$; *then* F *is* ACG($\mathbf{K}$-*sense*) *in* I *and* $\mathrm{p}ADF(t) = f(t)$ *in* I.

Proof. Let E be a closed set containing points of I; then by Theorem 4.3 there is an interval $J \subseteq I$ containing points of E such that

(i) $f\chi_E$ is $\mathfrak{L}$-integrable on J,

(ii) the series $\sum F(J_k \cap (:))$ converges unconditionally in $\mathfrak{C}_J(\mathbf{K})$ (where $\{J_k\}$ are the intervals in $J\backslash E$), and

(iii) $F(J') = \mathfrak{L}(f\chi_E, J') + \sum F(J_k \cap J')$ for all $J' \subseteq J$.

Firstly it can be verified, somewhat as in the previous theorem, that both the functions $\mathfrak{L}(f\chi_E, (:))$ and $\sum F(J_k \cap (:))$ are AC($\mathbf{K}$-sense) on E in J and hence that F is AC($\mathbf{K}$-sense) on E in J; by Lemma 5.1 then F is ACG($\mathbf{K}$-sense) in I.

As before construct the function g where $g(t) = f(t)$ $(t \in E \cap J)$ and $g(t) = F(J_k)/m(J_k)$ $(t \in J_k)$. The series $\sum F(J_k)$ converges unconditionally in $\mathbf{B}$ but not necessarily absolutely; thus although g may not be $\mathfrak{L}$-integrable on J it is certainly true that $x^*(g)$ is summable on J for each $x^* \in \mathbf{B}^*$ and moreover that $x^*(F(J')) = \int_{J'} x^*(g(t))\, dt$ for every interval $J' \subseteq J$ with end-points in E.

The arguments of the proof of Theorem 5.4 show that

$$AD[x^*(F)](t) = x^*(f(t)) \text{ a.e. in } I;$$

by the definition of approximate pseudo-derivative (see § 1) then $\mathrm{p}ADF(t) = f(t)$ in I and the theorem is proved.

It should be remarked that the concepts ACG$_s$($\mathbf{K}$-sense) and ACG$_*$($\mathbf{K}$-sense) do not in general characterize completely, as in the classical case, the indefinite $\mathfrak{D}_s(\mathbf{K})$- and $\mathfrak{D}_*(\mathbf{K})$-integrals. In fact for general Banach spaces $\mathbf{B}$ there may exist functions which are sAC in an interval and which fail to possess even strong approximate derivatives. The interested reader is referred to (**1**) and (**7**) for discussions of the 'property D' on a Banach space $\mathbf{B}$ which does permit such characterizations. If $\mathbf{B}$ has the 'property D' then a modification of a theorem of Alexiewicz ((**1**) Theorem 1, p. 102) shows that every function which is [sACG] in an interval is strongly approximately differentiable almost everywhere in that interval. In such cases, then, Lemma 5.2 and

Theorem 5.3 show that every function which is ACG_s(**K**-sense) or ACG_*(**K**-sense) in an interval is an indefinite $\mathfrak{D}_s(\mathbf{K})$- or $\mathfrak{D}_*(\mathbf{K})$-integral respectively.

6. Examples

As a typical example of the theory of this paper we shall investigate in our context a scale of integrals between the Lebesgue integral and the Denjoy integrals. Similar integrals were first introduced, with applications to trigonometric series, in (**2**) based on the concept of '*p*th power variation' of Norbert Wiener and L. C. Young.

Definition 10. The functions $\omega_p{}^r$ and $\omega_p{}^l$ are defined by the relations $\omega_p{}^r(F,I) = \omega_p{}^l(F,I) = \|(F,I)\|_p$ where, if $1 \leqslant p < +\infty$,

$$\|(F,I)\|_p = \sup\{[\sum |F(J_k \cap I)|^p]^{1/p}\}$$

the supremum being taken with respect to all finite sequences $\{J_k\}$ of disjoint intervals, and, if $p = \infty$,

$$\|(F,I)\|_\infty = \sup\{|F(J \cap I)|: \text{all intervals } J\}.$$

The pair of functions $\{\omega_p{}^r, \omega_p{}^l\}$ is denoted by $\mathbf{V}_p$.

Lemma 6.1. *The pair* $\mathbf{V}_p$ *satisfies axioms* I–VI *for all* p $(1 \leqslant p \leqslant +\infty)$.

The proof offers no difficulty and is omitted; it should be noticed that $V(F,I) = \|(F,I)\|_1$ and that $\|(F,I)\|_q \leqslant \|(F,I)\|_p$ if $1 \leqslant p \leqslant q \leqslant +\infty$ (Jensen's inequality).

Thus the integrals $\mathfrak{D}(\mathbf{V}_p)$, $\mathfrak{D}_s(\mathbf{V}_p)$, and $\mathfrak{D}_*(\mathbf{V}_p)$ are each defined and the theoretical apparatus of the previous sections applies to them. The first property we state is an immediate consequence of the Jensen inequality.

Theorem 6.2. *The* $\mathfrak{D}(\mathbf{V}_p)$-, $\mathfrak{D}_s(\mathbf{V}_p)$-, *and* $\mathfrak{D}_*(\mathbf{V}_p)$-*integrals all increase in generality for increasing* p.

One simplifying feature of the present example is the fact that the topological vector spaces $\mathfrak{B}_I(\mathbf{V}_p)$ and $\mathfrak{C}_I(\mathbf{V}_p)$ which play a fundamental role in the theory are here Banach spaces.

Theorem 6.3. *The spaces* $\mathfrak{B}_I(\mathbf{V}_p)$ *and* $\mathfrak{C}_I(\mathbf{V}_p)$ *are (equivalent to) Banach spaces with the norm* $F \to \|(F,I)\|_p$ *for each* p, $1 \leqslant p \leqslant +\infty$.

Proof. The arguments are standard and are omitted.

We continue by proving several theorems which permit a simplification of the conditions defining our integrals.

THEOREM 6.4. *Suppose that* $1 \leqslant p < +\infty$, *that* $\{J_k\}$ *is a sequence of disjoint intervals and that, for each* k, $F(J_k \cap (:)) \in \mathfrak{B}_I(\mathbf{V}_p)$. *Then* $\sum F(J_k \cap (:))$ *converges unconditionally in* $\mathfrak{B}_I(\mathbf{V}_p)$ *if and only if*

$$\sum \| (F, J_k \cap I) \|_p{}^p < +\infty.$$

Proof. It may be proved directly that, for each $I' \subseteq I$ and any finite set of indices π,

$$\left\| \left(\sum_{k \in \pi} F(J_k \cap (:)), I' \right) \right\|_p^p = \sum_{k \in \pi} \| (F, J_k \cap I') \|_p{}^p$$

from which the theorem obviously follows.

COROLLARY 6.5. *The* $\mathfrak{D}(\mathbf{V}_1)$-, $\mathfrak{D}_s(\mathbf{V}_1)$- *and* $\mathfrak{D}_*(\mathbf{V}_1)$-*integrals are equivalent.*

Proof. As a result of the theorem the three conditions defining these integrals coincide.

THEOREM 6.6. *Suppose that* $\{J_k\}$ *is a sequence of disjoint intervals and that, for each* k, $F(J_k \cap (:)) \in \mathfrak{B}_I(\mathbf{V}_\infty)$. *Then* $\sum F(J_k \cap (:))$ *converges unconditionally in* $\mathfrak{B}_I(\mathbf{V}_\infty)$ *if and only if* $\sum F(J_k \cap I)$ *converges unconditionally in* $\mathbf{B}$ *and* $\lim_{k \to \infty} \| (F, J_k \cap I) \|_\infty = 0$.

Proof. It is not difficult to see that the conditions are necessary; to show that they are also sufficient suppose that they are fulfilled. Then we may define a function G by $G(J) = \sum F(J_k \cap J)$ $(J \subseteq I)$; let π be an arbitrary finite set of indices, let $I' \subseteq I$ and note that

$$\left\| \left(G - \sum_{k \in \pi} F(J_k \cap (:)), I' \right) \right\|_\infty \leqslant \sup \left\{ \left| \sum_{k \notin \pi} F(J_k \cap J) \right| : J \subseteq I' \right\}$$

$$\leqslant \sup_\sigma \left\{ \left| \sum_{k \in \sigma} F(J_k \cap I') \right| \right\} + \sup \{ \| (F, J_k \cap I') \|_\infty : k \notin \pi \},$$

where σ denotes an arbitrary set of indices disjoint from π. But the right-hand side of this expression can be made as small as we please for arbitrary $I' \subseteq I$ and sufficiently large choices of π; in fact in the sense of the directed set $\{\pi, \subseteq\}$ we have $\lim \sum_{k \in \pi} F(J_k \cap (:)) = G$ in $\mathfrak{B}_I(\mathbf{V}_\infty)$ which is equivalent to the assertion of the theorem.

By comparison with ((**1**) 117) we now obtain

COROLLARY 6.7. *The* $\mathfrak{D}_s(\mathbf{V}_\infty)$ *integral is equivalent to the Denjoy–Bochner integral.*

When the space $\mathbf{B}$ is simply the space $\mathbf{R}$ of real numbers both the $\mathfrak{D}_s(\mathbf{V}_\infty)$- and the $\mathfrak{D}(\mathbf{V}_\infty)$-integrals reduce to the Denjoy–Khintchine integral: Alexiewicz (**1**) has not considered a Banach space version of the Denjoy–Perron integral but, in view of Theorem 6.6, it is apparent that the $\mathfrak{D}_*(\mathbf{V}_\infty)$-integral may be taken as such a generalization. At the other end of the scale the extension procedures are trivial; we shall state this without proof.

THEOREM 6.8. *The integrals $\mathfrak{D}(\mathbf{V}_1)$, $\mathfrak{D}_s(\mathbf{V}_1)$, and $\mathfrak{D}_*(\mathbf{V}_1)$ are equivalent to the integral $\mathfrak{L}$.*

THEOREM 6.9. *Let $1 \leqslant p < +\infty$. A necessary and sufficient condition that a function f be $\mathfrak{D}(\mathbf{V}_p)$-integrable (resp. $\mathfrak{D}_s(\mathbf{V}_p)$-integrable) on an interval I is that f be $\mathfrak{D}(\mathbf{V}_\infty)$-integrable (resp. $\mathfrak{D}_s(\mathbf{V}_\infty)$-integrable) on I and the function $\mathfrak{D}(\mathbf{V}_\infty)(f,(:))$ be in $\mathfrak{B}_I(\mathbf{V}_p)$.*

Proof. The stated condition is necessary since (by Theorem 6.2) the $\mathfrak{D}(\mathbf{V}_\infty)$-integral always includes the $\mathfrak{D}(\mathbf{V}_p)$-integral and an indefinite $\mathfrak{D}(\mathbf{V}_p)$-integral is certainly in the space $\mathfrak{B}_I(\mathbf{V}_p)$. Conversely suppose that $(f,I) \in \operatorname{dom}\mathfrak{D}(\mathbf{V}_\infty)$ and that $F = \mathfrak{D}(\mathbf{V}_\infty)(f,(:)) \in \mathfrak{B}_I(\mathbf{V}_p)$. Let $\mathfrak{F}$ denote the class of intervals $J \subseteq I$ such that $(f,J) \in \operatorname{dom}\mathfrak{D}(\mathbf{V}_p)$. The family $\mathfrak{F}$ has the properties (i), (ii), and (iii) of (2.1): the first two properties are trivial while the latter requires some justification. This follows from the fact that $\mathfrak{D}(\mathbf{V}_p)^{C(\mathbf{V}_p)} = \mathfrak{D}(\mathbf{V}_p)$ and the following assertion which we shall prove.

(6.10) *Suppose that $F \in \mathfrak{C}_J(\mathbf{V}_\infty) \cap \mathfrak{B}_J(\mathbf{V}_p)$ and that $F(J_0 \cap (:)) \in \mathfrak{C}_J(\mathbf{V}_p)$ for every interval J_0 with $\bar{J}_0 \subseteq J$. Then $F \in \mathfrak{C}_J(\mathbf{V}_p)$.*

To prove this let $J = (a,b)$, choose a doubly infinite sequence $\{\varepsilon_k\}$ so that $a < \dots < \varepsilon_{-1} < \varepsilon_0 < \varepsilon_1 < \dots < b$ with $\lim_{k\to-\infty} \varepsilon_k = a$ and $\lim_{k\to+\infty} \varepsilon_k = b$, and write $J_k = (\varepsilon_k, \varepsilon_{k+1})$. By hypothesis F is $\mathbf{V}_\infty$-continuous on J (i.e. strongly continuous in the classical sense) so that $F(J') = \sum_{-\infty}^{+\infty} F(J_k \cap J')$ for every interval $J' \subseteq J$. But $\sum \|(F,J_k)\|_p{}^p \leqslant \|(F,J)\|_p{}^p < +\infty$ since $F \in \mathfrak{B}_J(\mathbf{V}_p)$ and hence, by Theorem 6.4, the series $\sum F(J_k \cap (:))$ converges unconditionally (to F) in the space $\mathfrak{B}_J(\mathbf{V}_p)$. By hypothesis $F(J_k \cap (:)) \in \mathfrak{C}_J(\mathbf{V}_p)$ for each k and, since $\mathfrak{C}_J(\mathbf{V}_p)$ is closed in $\mathfrak{B}_J(\mathbf{V}_p)$, it follows that the sum of the series, F, is also in that space as required.

Returning to the proof of the theorem we see that the proof is completed if $I \in \mathfrak{F}$; let E be the complement of the set $\bigcup\{J : J \in \mathfrak{F}\}$ and note, by (2.1), that if $E \cap I = \emptyset$ the theorem is proved. Suppose on the contrary that E contains points of I; then, by Theorem 4.3, since E is closed there is an interval $J \subseteq I$ containing points of E such that

(i) $(f\chi_E, J) \in \operatorname{dom}\mathfrak{L}$,

(ii) $\sum F(J_k \cap (:))$ converges unconditionally in $\mathfrak{C}_J(\mathbf{V}_\infty)$, and

(iii) $F(J') = \mathfrak{L}(f\chi_E, J') + \sum F(J_k \cap J')$ $(J' \subseteq J)$ where $\{J_k\}$ are the intervals in $J \setminus E$.

But $F \in \mathfrak{B}_I(\mathbf{V}_p)$ so that $\sum \| (F, J_k) \|_p{}^p \leqslant \| (F, I) \|_p{}^p < +\infty$ and hence, by Theorem 6.4, the series $\sum F(J_k \cap (:))$ converges unconditionally in $\mathfrak{B}_I(\mathbf{V}_p)$. Thus (since by (2.1) each interval $J_k \in \mathfrak{F}$) f is $\mathfrak{D}(\mathbf{V}_p)$-integrable on each interval J_k, $f\chi_E$ is $\mathfrak{L}$-integrable on J and the series $\sum F(J_k \cap (:))$ converges unconditionally in $\mathfrak{B}_J(\mathbf{V}_p)$; by definition then f is $\mathfrak{D}(\mathbf{V}_p)$-integrable on J also which contradicts the fact that $J \cap E \neq \emptyset$ and completes the proof of the theorem.

This theorem combined with ((**2**) Theorem 2.5, p. 212) will permit the interested reader to establish the relationship between the $\mathfrak{D}(\mathbf{V}_p)$-integrals and the Burkill–Gehring integrals.

We shall conclude with a brief discussion of the Cesàro–Denjoy (Cesàro–Perron) integrals of Sargent (**11**). Throughout the following we must simplify by requiring that $\mathbf{B} = \mathbf{R}$.

Definition 11. The functions ${}_\lambda\omega^r$ and ${}_\lambda\omega^l$ are defined by the relations

(i) ${}_0\omega^r(F, (a,b)) = \sup\limits_{a<t<b} |F(a,t)|$,

(ii) ${}_0\omega^l(F, (a,b)) = \sup\limits_{a<t<b} |F(t,b)|$,

and, for $\lambda = 1, 2, \ldots$,

(iii) ${}_\lambda\omega^r(F, (a,b)) = \sup\limits_{a<t<b} \left| \lambda/(t-a)^\lambda \int_a^t F(a,\tau)(t-\tau)^{\lambda-1}\, d\tau \right|$,

(iv) ${}_\lambda\omega^l(F, (a,b)) = \sup\limits_{a<t<b} \left| \lambda/(b-t)^\lambda \int_t^b F(\tau,b)(t-\tau)^{\lambda-1}\, d\tau \right|$,

provided that in (iii) and (iv) the integrals exist in the sense of the Cesàro–Denjoy integral of order $\lambda - 1$; otherwise the left-hand side of the relation is defined to be ∞. The pair of functions $\{{}_\lambda\omega^r, {}_\lambda\omega^l\}$ is denoted by $\mathbf{C}_\lambda$.

Lemma 6.11. *The pair $\mathbf{C}_\lambda$ for $\lambda = 0, 1, \ldots$ satisfies axioms* I–VI.

Proof. Axioms I, II, III, and VI may be directly verified; axioms IV and V follow, with some manipulation, from several fundamental inequalities of Sargent ((**11**) 216). We omit the details.

Theorem 6.12. *The $\mathfrak{D}_*(\mathbf{C}_\lambda)$-integral is equivalent to the Cesàro–Denjoy integral of order λ for each non-negative integer λ.*

Proof. This follows directly from Theorem 5.4 and the definitions in (**11**).

It should also be remarked that the generalized mean-continuous integrals of Ellis (**3**) may be placed within the present context: Sargent

(**12**) has proved that a function which is generalized mean-continuous of order λ (see (**3**)) is necessarily integrable in the Cesàro–Denjoy sense of order $\lambda-1$ and hence may be considered to be $\mathbf{C}_\lambda$-continuous in our notation. Thus the GM_λ-integrals of (**3**) are $\mathbf{C}_\lambda$-integrals in the sense of § 3 and subject to the present analysis. Certainly the GM_λ-integrals include the $\mathfrak{D}(\mathbf{C}_\lambda)$-integrals but, except for the special case $\lambda = 0$, we are unable at present to assert the converse.

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