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A SYMMETRIC COVERING THEOREM

By a symmetric full cover of the real line is meant a collection $\mathcal{S}$ of closed intervals with the property that for every real x there is a $\delta(x) > 0$ so that $[x-h, x+h] \in \mathcal{S}$ for every $0 < h < \delta(x)$. It has been shown[1] that for such a collection $\mathcal{S}$ there is a closed denumerable set $C \subset (0,\infty)$ so that $\mathcal{S}$ contains a partition of every interval $[-x, x]$ with $x \notin C$. The simplicity and utility of this result may have led some to overlook an extension that is on occasion more useful.

THEOREM. *Let $\mathcal{S}$ be a symmetric full cover on the real line. Then there is a denumerable set N so that $\mathcal{S}$ contains a partition of every interval neither of whose endpoints belongs to N.*

For every real x choose $\delta(x) > 0$ so that $[x-h, x+h] \in \mathcal{S}$ for every $0 < h < \delta(x)$. The proof follows in three simple steps.

(1) For every x there is a denumerable set $C_x \subset (x, \infty)$ so that $\mathcal{S}$ contains a partition of $[x-w, x+w]$ for every $x + w \notin C_x$. This is just the result already mentioned[2].

(2) For every nonzero h there is a denumerable set T_h so that $\mathcal{S}$ contains a partition of $[x, x+h]$ (or $[x+h, x]$ if $h < 0$) for every $x \notin T_h$. For $h > 0$ write

$$T_h = \bigcup_{r \in Q} (2r - C_r) \cup (C_{r+h/2} - h)$$

where the union is taken over all rationals. If $x \notin T_h$, then choose a rational number $r \in (x, x + h/2)$. Using the centers r and $r + h/2$ we see that $\mathcal{S}$ contains a partition of $[x, 2r - x]$ because $x \notin (2r - C_r)$ and $\mathcal{S}$ contains a partition of $[2r - x, x + h]$ because $x \notin (C_{r+h/2} - h)$. This gives a partition of $[x, x+h]$ as required. A similar argument works for $h < 0$.

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[1] B. S. Thomson, *Real Analysis Exchange*, 6 (1980/81), 77–93
[2] *op. cit.*

(3) There is a denumerable set N so that S contains a partition of $[x,y]$ if $x,y \notin N$. Let $N = \bigcup_{r\in\mathbf{Q}} T_r$ where the union is taken over all nonzero rationals. If neither x nor y belongs to N, then choose a rational $s > 0$ so that $x + s \in (m - \delta(m), m)$ where $m = (x+y)/2$. Evidently S contains a partition of $[x, x+s]$ and of $[y-s, y]$; since it also contains the interval $[x+s, y-s]$ it contains a partition of $[x,y]$ as required. This completes the proof.

We hope that this covering theorem will not inhibit the discovery of other useful covering results. However we should part with the warning that a search for an approximate or "qualitative" analogue has a small trap: under the continuum hypothesis there is a nonmeasurable function f such that $\{t > 0 : f(x+t) \neq f(x-t)\}$ is denumerable for every x.

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