Real Analysis Exchange
Summer Symposium 2003, pp. 113–118

Brian S. Thomson, Simon Fraser University, B. C., Canada.
email: `thomson@cs.sfu.ca`

VBG Functions

1 Background–Classical

Let f be a continuous real function defined on the real line that is locally of bounded variation. Then, using the Jordan decomposition to obtain the total variation function of f, we can describe a measure V_f which expresses the total variation of f. This *total variation measure* V_f allows for many interconnections among the differentiation, integration and variational properties of f. The following are well known and should not need much comment:

$$V_f([a,b]) = V(f,[a,b]) \tag{1}$$

expresses the usual variation $V(f,[a,b])$ on an interval $[a,b]$ in terms of the measure;

$$V_f(E) = \int_E |f'(t)|\,dt \tag{2}$$

for any Borel set E at each point of which f has a finite derivative, serves to link the measure to differentiation properties of the function; and finally the de la Vallée Poussin decomposition

$$V_f(E) = \int_{E\cap D_1} |f'(t)|\,dt + V_f(E\cap D_2) \tag{3}$$

where E is a Borel set and D_1 and D_2 denote the (Borel) sets at which f has a finite or, respectively, an infinite derivative, serves to reveal the structure of the measure in terms of general differentiation properties of f.

In this one paragraph, stating classical properties of a real function, can be found also in a recognizable and concrete form the usual ingredients of a first course in measure theory: the Jordan, Hahn and Lebesgue decompositions and the Radon-Nikodym theorem.

If f fails to have bounded variation on some interval (for example most differentiable functions would fail in this way) then such a total variation measure

[0] A summary, more or less, of a talk given at the Real Analysis Summer Symposium 2003.

would have to be obtained in an entirely different manner. See for example the abstract of the talk given by Sebastian Lindner in this symposium. His measure has the feature that it assigns infinite measure at any point at which the function has unbounded variation; this is a reasonable feature for some studies, but not desirable in a measure that would carry the differentiation structure seen in (2) and (3).

For that we might turn to the variational ideas of the 1920's and 1930's studied by Denjoy, Khintchine, Lusin and Saks, which are fully developed in Saks treatise [3]. Recall one writes

Definition 1.1

$$V(f,E) = \sup \sum_{i=1}^{n} |f(b_i) - f(a_i)| \tag{4}$$

and

$$V_*(f,E) = \sup \sum_{i=1}^{n} \sup_{a_i \le x_i \le y_i \le b_i} |f(y_i) - f(x_i)| \tag{5}$$

where the supremum is taken over sequences of nonoverlapping intervals $\{[a_i, b_i]\}$ *with endpoints in* E.

We say f is VBG on a set E if there is a countable partition of E into a sequence of sets $\{E_n\}$ for each of which $V(f, E_n) < \infty$. We say f is VBG$_*$ on a set E if there is a countable partition of E into a sequence of sets $\{E_n\}$ for each of which $V(f, E_n) < \infty$.

One often remembers these by the phrase "The VBG and VBG$_*$ functions are σ-finite analogues of BV functions." But these expressions are not measures. While $V_*(f([a,b]) = V(f,[a,b])$ is the usual variation on an interval there are no statements at all similar to (2) and (3).

These classes of functions play an important role in the study of derivatives and integrals on the real line, but the expressions $V_*(f,E)$ and $V(f,E)$ are unfortunate tools in expressing these ideas.

There is a simple and natural way to define a total variation measure V_f that preserves all of the features expressed in equations (1), (2), and (3) and moreover makes the proposed analogy above quite correct: VBG$_*$ functions are indeed those functions with a σ-finite total variation measure.

The VBG functions have not received a similar treatment and it is that that now motivates us. Is there an appropriate analog of a total variation measure expressing the notion of this class of functions and which preserves in some way the fundamental properties (1), (2), and (3)?

2 Variational Measures

Here is a shortened version of the usual definition, re-expressed in a way that should not be too unfamiliar to most who use these concepts.

By a (set-valued) gage we mean a function $\delta : \mathbb{R} \to 2^{\mathbb{R}}$.

Definition 2.1 *For any real function, any set* E *of real numbers and any gage* δ *we write*

$$V(f,E;\delta) = \sup \sum_{i=1}^{n} |f(y_i) - f(x_i)| \tag{6}$$

where the supremum is taken over all nonoverlapping collections of intervals $\{[x_i, y_i]\}$ *with* $x_i \in E$, $y_i \ne x_i$ *and* $y_i \in \delta(x_i)$. *(We allow either* $x_i < y_i$ *or* $y_i < x_i$ *so that the interval* $[x_i, y_i]$ *can also mean* $[y_i, x_i]$ *in the usual notation for intervals.)*

Define a gage δ to be an *ordinary gage* if $\delta(x)$ is a neighborhood of x for each real x. The *total variation measure* V_f is defined by

$$V_f(E) = \inf_{\delta} V(f,E;\delta) \tag{7}$$

where the infimum is taken over all ordinary gages δ.

The set function V_f is what is usually called an outer measure or metric outer measure or Carathéodory measure. Restricted to the Borel sets or to its measurable sets it is a measure in the usual sense of that term. Properties (1), (2) continue to hold in general and property (3) remains valid in the wider class of continuous VBG$_*$ functions. VBG$_*$ functions are exactly those functions with a σ-finite total variation measure. These properties, and many more, clearly suggest that this is the "right" way to develop the variational ideas of Denjoy, Lusin, Khintchine and Saks.

We propose the following two variants on the total variation measure associated with a function f. Each is an attempt to mimic some feature of the VBG theory. The first idea flows from the fact that the VBG concept is (as is apparent from the version in Saks [3]) tightly linked to the approximate derivative and the second idea is to mimic directly the structure of the VBG definition.

Definition 2.2 *A gage* δ *is said to be an* approximate gage *if for each* x *the set* $\delta(x)$ *has inner Lebesgue density* 1 *at* x.

Definition 2.3 *A gage* δ *is said to be an* T-gage *if there is a countable collection of sets* $\mathcal{X} = \{X_1, X_2, X_3, \dots\}$ *covering the real line and a numerical gage* η *so that*

$$(x - \eta(x), x + \eta(x)) \cap X \subset \delta(x)$$

for every $X \in \mathcal{X}$ *and every* $x \in X$.

Definition 2.4 *The* approximate variation *of a real function* f *on a set* E *is defined by*

$$V_f^{app}(E) = \inf_{\delta} V(f, E; \delta) \tag{8}$$

where the infimum is taken over all approximate gages δ.

Definition 2.5 *The* T-variation *of a real function* f *on a set* E *is defined by*

$$V_f^T(E) = \inf_{\delta} V(f, E; \delta) \tag{9}$$

where the infimum is taken over all T-gages δ.

The approximate version has received some attention. The T-variation is named after Tolstov [2] who worked on a related problem in the setting of Perron-type integrals and the Denjoy-Khintchine integral. It is partially related to this structure. In [1, Exercise 42.9, p. 222] there was a similar attempt at using Tolstov's ideas in this setting but the author, in a later paper, pointed out the error.

It is easy to check that V_f^{app} is a metric outer measure (using the same arguments as for V_f); that V_f^T is also a metric outer measure requires somewhat different methods.

We might expect, naively, since the variational measure V_f^{app} is so intimately connected to the process of approximate differentiation and since the concept VBG plays such a key role in the Saks [3] development of the approximate derivative, that it would follow that there is a close relation between the measure and the VBG concept. Indeed there are connections but not as intimate as those connections between the ordinary variation V_f and the VBG_* concept: for example, recall that a continuous function f is VBG_* on a set E precisely when V_f is σ-finite on E.

Theorem 2.6 *Let* f *be a real function and suppose that* V_f^{app} *is* σ*-finite on a set* E. *Then* f *is VBG on* E.

But the converse fails, even for simple examples. Indeed there is even a continuous VBG (in fact ACG) function f so that V_f^{app} is not σ-finite. A similar example and for a similar purpose was given by Tolstov.

On the other hand the T-variation does indeed characterize the VBG concept as the next theorem shows.

Theorem 2.7 *Let* f *be a real function. Then* f *is VBG on a set* E *if and only if* V_f^T *is* σ*-finite on* E.

But this is of little interest if the variational measure has no merits or other applications of its own. The problem here is not merely to characterize the class of VBG functions by a measure-theoretic statement, but to find a possibly useful tool for their study. Our test of that usefulness is whether there are statements analogous to (1), (2) and (3) in the introduction.

One might begin to worry if one notes the obvious fact that if any function f with countable range is VBG and so V_f^T is σ-finite as we just learned. But for such a function V_f^T vanishes on every set and so carries no variational information about the function f and certainly no statement such as (1) need be true.

For continuous functions f the variational measure V_f^T does compute the ordinary variation on an interval.

Theorem 2.8 *Let* f *be a continuous real function and* $[a,b]$ *an arbitrary closed interval. Then*

$$V_f^T([a,b]) = V(f, [a,b]).$$

A development of the analogues to (2) and (3) requires introducing an appropriate generalized derivative arising from this structure. Space allowed here, however, does not allow us to begin.

References

[1] R. Henstock, *Linear Analysis*, Butterworth & Co. Ltd., (1967).

[2] G. Tolstov, *Sur l'intégrale de Perron*, Mat. Sb., **5, (47)** 647–660.

[3] S. Saks, *Theory of the Integral*, Dover, (1937).