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MONOTONICITY THEOREMS

Recently in this Exchange Professor Bruckner [3] has raised the problem of finding some unified approach to the subject of monotonicity theorems. He has split the attack on this problem into two categories: reduction theorems and abstract theorems. There are now some successful and interesting reduction theorems that do give much insight into the nature of a number of monotonicity results. Most prominent among these is due to Bruckner himself and asserts, roughly, that any monotonicity theorem provable for the class of functions that are continuous and of bounded variation is extendable to the class of Darboux-Baire 1 functions. For a discussion of this (and a more precise formulation) and of a reduction theorem due to O'Malley and Weil the reader is referred to [3, pp. 28-33].

The abstract approach to the monotonicity problem has so far yielded less significant results. The most promising, as Bruckner [3, pp. 33-34] points out, lies in the study of the selective derivatives introduced by O'Malley but it may be that this class of derivatives is too specialized to reveal much about the structure of many monotonicity theorems. The most obvious abstract approach would be to invoke the machinery of abstract differentiation theory and somehow bring it to bear on this problem.

Abstract differentiation theory has been under study now for nearly half a century and has achieved a certain maturity. It has clarified many of the difficulties of the differentiation theory of integrals in Euclidean space and most recently, with the help of the lifting theorem, it has placed the Radon-Nikodym theorem for abstract measure spaces into a proper differentiation setting. The best and fastest introduction to the field would be to consult Bruckner [1] and then work through the bibliography there.

Unfortunately though not much of this theory has any bearing on the monotonicity problem. The reason for this is that the abstract theory has mainly centered its attention on the Vitali covering theorem and the differentiation theory for integrals. Thus one encounters properties labelled as strong Vitali, weak Vitali, ψ-Vitali, p-Vitali as well as properties that are motivated by some proof of the Vitali theorem and one finds the Hardy-Littlewood operators playing a major role in the theory. But on the real line, while these concepts make sense they do nothing to distinguish between various differentiation bases. For example the Vitali covering theorem will hold for almost any differentiation basis on the real line and with respect to any reasonable measure (see de Guzmán [5, p. 27]). As it stands then even though all of the problems of differentiation theory on the real line can be expressed in the language of the abstract theory, that theory contains no tools with which to solve them.

Nonetheless we can apply the general program of the abstract theory: find "geometric" properties of an abstract differentiation basis that can be used to unify monotonicity theorems. These properties will mainly be special and peculiar to the real line setting and so we must consider that what we are doing here is merely a special and peculiar chapter in the larger abstract theory of differentiation. This article contains an expository account of our attempts to initiate this theory in [10]. Abstract theories can be quite formidable and forbidding in their full-dress formal presentation as anyone familiar with the current literature of abstract differentiation theory will appreciate; even though the basic ideas may be simple and elegant the notation and terminology alone may be enough to put off all but the most determined reader. It is hoped that this informal and motivated account will clarify our intentions in this study and perhaps encourage others to follow this program.

1. <u>The program</u>. Let us accept that a proper attack on the monotonicity problem would be to investigate the subject within an abstract differentiation theory. Indeed much of the theory

of generalized derivatives on the real line might be best developed within such a framework. The simplest account of such a theory can be given as follows:

- let $[a,b]$ be a fixed interval.
- let $\mathcal{I}$ denote the family of all closed subintervals of $[a,b]$.
- let $B(x)$ be, for each $x \in [a,b]$, a filterbase* on $\mathcal{I}$.

This object B, however one wishes to view it, is our abstract differentiation basis and gives rise to derivatives and derivates in an obvious manner:

if f is a real-valued function on $[a,b]$ then the extreme derivates of f with respect to the differentiation basis B are

$$\underline{D}_B f(x) = \sup_{\beta \in B(x)} \inf\{f(I)/|I| : I \in \beta\}$$

and

$$\overline{D}_B f(x) = \inf_{\beta \in B(x)} \sup\{f(I)/|I| : I \in \beta\} .$$

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*By a filterbase on $\mathcal{I}$ we mean that each $\beta \in B(x)$ is a nonempty subset of $\mathcal{I}$ and whenever β_1 and β_2 belong to $B(x)$ there must be in $B(x)$ an element $\beta_3 \subseteq \beta_1 \cap \beta_2$. The theory is also traditionally developed using nets in place of filters; we have chosen this present viewpoint as being simpler but no less general.

Here of course we are writing $|I|$ for the length of the interval I and $f(I)$ for the usual increment of f on I.

It should be clear how to realize any number of generalized derivatives within the setting: the Dini derivatives, the symmetric extreme derivates, the approximate extreme derivates, etc. permit this characterization while the preponderant would not. The program is, simply stated, to find properties of the family $\{B(x) : x \in [a,b]\}$ that permit various assertions about the corresponding derivates to hold. For example consider the following "theorem":

A. If for every $x \in [a,b]$, $\underline{D}_B F(x) \geq 0$, then f must be nondecreasing on that interval.

This theorem has been proved to hold for the ordinary derivatives, the approximate derivates, the selective, and the qualitative derivates; it is not true for the Dini derivatives or the symmetric derivatives. What then is the geometry of this theorem? Is there some property shared by the differentiation bases and expressible strictly in terms of the filterbase $B(x)$ that makes this theorem true for some derivatives and false for others? A review of the proofs is not particularly revealing for they commonly employ indirect arguments and use special properties of the corresponding derivatives. Indeed there is no reason why this question must be answerable.

Fortunately there is a meaningful and simple abstract version of the above monotonicity result. The true nature of the theorem does lie within the geometry of the filterbase structure. For our purposes though this present formulation is particularly awkward and lacks much of the flexibility and notational ease of an essentially equivalent formulation.

The collection $\{B(x) : x \in [a,b]\}$ of filterbases on $\mathcal{I}$ can be better visualized as a single filterbase B on $\mathcal{I} \times [a,b]$. The extreme derivates are written then as

$$\underline{D}_B f(x) = \sup_{\beta \in B} \inf_{(I,x) \in \beta} f(I)/|I|$$

and

$$\overline{D}_B f(x) = \inf_{\beta \in B} \sup_{(I,x) \in \beta} f(I)/|I| .$$

This presentation allows us considerable notational advantage; more than that it is a genuine generalization of the traditional structure in that it allows us to discuss uniform derivatives as well as pointwise ones. Even more it permits us to tie in the theory with the theory of integration as developed in the works of Ralph Henstock [6]; thus various theories of generalized derivatives and generalized integrals emerge from the same structure and the traditional inter-relations that hold between the two subjects emerge in a natural and revealing way. Our interest here is restricted to the monotonicity problem and we shall ask of the reader only that he be prepared to accept as a rational approach to that problem the investigation of filterbase structures on $\mathcal{I} \times [a,b]$.

2. Abstract derivation theory. For the reasons outlined above we have chosen to base the theory on the following structures:

- $[a,b]$ is a fixed interval of real numbers.
- $\mathcal{I}$ is the family of all closed subintervals of $[a,b]$.
- $\mathcal{B}$ is a given nonempty collection of subsets of $\mathcal{I} \times [a,b]$.

We shall speak loosely of such a $\mathcal{B}$ as a derivation basis and define the extreme derivates for a function f relative to this basis by writing:

$$\underline{D}_{\mathcal{B}} f(x) = \sup_{\beta \in \mathcal{B}} \inf_{(I,x) \in \beta} f(I)/|I|$$

and

$$\overline{D}_{\mathcal{B}} f(x) = \inf_{\beta \in \mathcal{B}} \sup_{(I,x) \in \beta} f(I)/|I| .$$

The notion of an exact $\mathcal{B}$-derivative could be defined by the customary device of requiring the two extreme derivates to be equal. Much better in this setting is to say that g

is an exact B-derivative of f if for every $\varepsilon > 0$ there is a $\beta \in B$ with

$$|g(x) - f(I)/|I|| < \varepsilon$$

for every $(I,x) \in \beta$. Although we will not be using this idea in the sequel it is a most interesting generalization of many derivation ideas.

It is important to realize here that we are making no assumptions in advance on the collection B beyond the fact that it is some nonempty collection of subsets of $\mathcal{I} \times [a,b]$. Thus, for example, it is not necessary for the upper derivate $\overline{D}_B f(x)$ to exceed the lower derivate $\underline{D}_B f(x)$ nor for an exact B-derivative to be unique. In fact the theory is much more flexible if we avoid any a priori assumptions on B. The following examples illustrate the scope and intention of the theory.

(i) [trivial derivation basis]. Let $B = \{\emptyset\}$. Then invariably $\underline{D}_B f(x) = +\infty$, $\overline{D}_B f(x) = -\infty$, and every function g is an exact B-derivative of f.

(ii) [uniform derivation basis]. Let $\delta > 0$ and write β_δ as the collection of all interval-point pairs (I,x) with $I \in \mathcal{I}$, $x \in I$, and $|I| < \delta$. Define

$$U = \{\beta_\delta : \delta > 0\}.$$

A function g is an exact U-derivative of f if and only if g is the uniform derivative of f. The extreme U-derivates are the ordinary extreme derivates but this is not the best derivation basis in which to study them.

(iii) [ordinary derivation basis]. Let δ be an aribitrary positive function on $[a,b]$ and define, similarly to (ii) above,

$$\beta_\delta = \{(I,x) : I \in \mathcal{I} , x \in I, |I| < \delta(x)\}$$

and take D as the collection of all such β_δ for positive functions δ on $[a,b]$. Then $\underline{D}_D f(x)$ and $\overline{D}_D f(x)$ are the usual extreme derivates $\underline{f}'(x)$ and $\overline{f}'(x)$; g is an exact D-derivative of f if and only if $f'(x) = g(x)$ (finitely) everywhere on $[a,b]$.

(iv) [reversed ordinary derivation basis]. Suppose that to each $x \in [a,b]$ there is given a set $\eta(x) \subseteq [a,b]$ which has x as a point of accumulation; write

$$\beta_\eta = \{([x,y],x) : y \in \eta(x), y \neq x\}$$

using the usual convention that $[x,y]$ with $y < x$ means $[y,x]$. Let R be the collection of all such β_η for any choice of functions η having the above stated property.

Then $\underline{D}_R f(x) = \overline{f}'(x)$, $\overline{D}_R f(x) = \underline{f}'(x)$, and g is an exact R-derivative of f if and only if $g(x)$ is a finite derived number of f at x for every $x \in [a,b]$.

The device used here can be applied to any derivation basis to effectively reverse the roles of the upper and the lower derivates, and to change an exact derivative to a derived number in some sense. This is closely related, of course, to the notion of a Vitali cover or B-fine cover in the language of abstract differentiation theory.

(v) [approximate derivation basis]. Suppose that $0 \leqq \rho \leqq 1$ and $0 \leqq \lambda \leqq 1$. Let $\eta(x)$ for any $x \in [a,b]$ denote a measurable set that has right density at x exceeding ρ (unless $\rho = 1$ in which case $\eta(x)$ has right density equal to 1 at x) and left density at x exceeding λ (unless $\lambda = 1$ in which case $\eta(x)$ has left density equal to 1 at x). Define for such a function η,

$$\beta_\eta = \{([y,z],x) : [y,z] \in \mathcal{I},\ y \leqq x \leqq z, \text{ and } y,z \in \eta(x)\} .$$

Let $A^{(\rho,\lambda)}$ denote the collection of such subsets β_η of $\mathcal{I} \times [a,b]$ for functions η having the above stated properties for a given pair ρ and λ.

For measurable functions these derivation bases can be used to study the extreme approximate derivates and the extreme preponderant derivates; $A^{(1,1)}$ gives the usual approximate derivatives (at least for measurable functions), and $A^{(1/2,1/2)}$ the usual preponderant ones. In general the bases $A^{(\rho,\lambda)}$ with $\rho + \lambda \geqq 1$ share most of the important properties of the approximate and preponderant bases.

These examples should suffice for the reader to place any of a number of generalized derivatives within this setting. In fact the scheme is somewhat wider than even these examples might indicate; as an indication we illustrate with a further example of a slightly different type.

(vi) For each $x \in [a,b)$ let $\eta(x)$ be a nonempty subset of $(x,b]$. Write

$$\beta_\eta = \{([x,y],x) : y \in \eta(x)\}$$

and

$$B = \{\beta_\eta : \text{all such functions } \eta\} .$$

For this derivation basis B the following "monotonicity" theorem is easily proved:

if f is continuous on $[a,b]$ and $\underline{D}_B f(x) \geqq 0$ everywhere then $f(b) \geqq f(a)$.

This places in a familiar form a theorem that at first sight would not appear to have any relation to a derivation result:

if f is continuous on $[a,b]$ and for each $x \in [a,b)$

$$\sup\{\frac{f(y) - f(x)}{y - x} : x < y \leq b\} \geq 0 \text{ then } f(b) \geq f(a).$$

We proceed to a study of the geometry of this structure B consisting of subsets of $\mathcal{I} \times [a,b]$. There are a number of elementary properties of such structures that can be used to prove monotonicity theorems and which unify some classical theorems. We need the following notation to simplify the assertions:

if β is a subset of $\mathcal{I} \times [a,b]$ and $X \subseteq [a,b]$ then

$$\beta[X] = \{(I,x) \in \beta : x \in X\} \text{ and } \beta(X) = \{(I,x) \in \beta : I \subseteq X\}.$$

Thus $\beta[X]$ just picks out from β those pairs (I,x) with the point x in the set X and $\beta(X)$ just picks out those pairs with the interval contained in X.

There are only five properties needed in the theory.

(1) <u>filtering down.</u> B is said to be filtering down if for every pair β_1 and β_2 from B there is a β_3 in B with $\beta_3 \subseteq \beta_1 \cap \beta_2$.

This is a natural and obvious property shared by most important derivation bases. Examples (i), (ii) and (iii) have this property, while (iv) does not. The basis $A^{(1,1)}$ is filtering down but in general the bases $A^{(\rho,\lambda)}$ are not.

(2) <u>pointwise character.</u> B is said to have pointwise character if for every family $\{\beta_x : x \in [a,b]\} \subseteq B$ there is a $\beta \in B$ with

$$\beta[\{x\}] \subseteq \beta_x$$

for each $x \in [a,b]$.

At first sight this property may seen formidably abstract; a moment's reflection should show though that it expresses only the requirement that the family B be characterized by its pointwise behaviour. Most derivatives (but not uniform derivatives) are defined locally and this property merely formalizes this requirement. For example a statement that $\underline{D}_B f(x) > g(x)$ everywhere in X means that there exists for each $x \in X$ an element $\beta_x \in B$ with $f(I) > g(x)|I|$ for every $(I,x) \in \beta_x$. If B is of pointwise character then this allows the choice of a $\beta \in B$ such that this same inequality holds for all $(I,x) \in \beta$ and any $x \in X$. Only example (ii), the uniform derivation basis, is not of pointwise character.

The terminology is lifted from McShane [7] in a different but related context; Henstock [6] uses <u>decomposable</u> for an equivalent idea, and exploits a weaker concept in his integration theory.

(3) finer than the topology. B is finer than the topology if for every $\beta_0 \in B$ and every open set $G \subseteq [a,b]$ there is a $\beta \in B$ with

$$\beta[G] \subseteq \beta_0[G].$$

The uniform derivation basis (example (ii)) does not have this property, nor does the peculiar example (v), but all genuine derivatives should have this property. It expresses only that the limit of the quotient $f(I)/|I|$ is taken somehow in terms of smaller and smaller neighbourhoods of the point x.

(4) partitioning property. B is said to have the partitioning property if for every interval $J \in \mathcal{I}$ and every $\beta \in B$ there is a pointed-partition $\pi \subseteq \beta$ of J; i.e.,

$$\pi = \{(I_i, x_i) : i = 1,2,\ldots,n\},$$

I_i and I_j do not overlap if $i \neq j$, and $\bigcup_{i=1}^{n} I_i = J$.

Certainly the uniform derivation basis, example (ii), has this property for it is exactly the setting for the Riemann integral. It is rather more surprising that this same property is shared by the bases for the ordinary (iii)), the approximate and preponderant (i.e., example (iv) with $\rho + \lambda \geq 1$), the selective, and the qualitative derivatives. It represents the simplest and most interesting geometric property to emerge in this general theory; it plays, of course, a key role in the general theory of Riemann-type integrals.

(5) Young decomposition. $\mathcal{B}$ is said to permit a Young decomposition if for every $X \subseteq [a,b]$ and for every $\beta \in \mathcal{B}$ there is a sequence of sets $\{X_n\}$ covering X such that $\beta[X_n]$ contains a pointed-partition of every interval with endpoints in X_n.

This property is admittedly more technical than the preceding and a little harder to motivate. It is an abstraction of a common device in derivation theory; for example the decomposition in the proof of Theorem 4.1 in [2, p. 63] or Theorem 10.8 in [9, p. 237] is precisely of this type. We have chosen to label it with the name of an illustrious family of mathematicians because it is the key property in establishing a generalization of two theorems for the Dini derivatives established by W.M. Young and G.C. Young in 1908 and 1912. The abstract versions are:

I. [Theorem of G.C. Young] If $\mathcal{B}_1$ and $\mathcal{B}_2$ are families of subsets of $\mathcal{I} \times [a,b]$ both of which have the Young decomposition property and are of pointwise character then, with at most countably many exceptions,

$$\overline{D}_{\mathcal{B}_1} F(x) \geq \underline{D}_{\mathcal{B}_2} f(x) .$$

II. [Theorem of W.H. Young] If B_1 and B_2 are families of subsets of $I \times [a,b]$ both of which are finer than the topology, filtering down, of pointwise character, and have the Young decomposition property then for any continuous function f,

$$\overline{D}_{B_1} f(x) = \overline{D}_{B_2} f(x) \quad \text{and} \quad \underline{D}_{B_1} f(x) = \underline{D}_{B_2} f(x)$$

for all but a first category subset of $[a,b]$.

If B_1 and B_2 are taken as the derivation bases for the left and right Dini derivatives these theorems are exactly the classical theorems of the two Youngs.

These five properties are enough now to start an attack on the monotonicity problem and in the next section we will show how they can be employed to give an indication of why some monotonicity theorems work.

3. Elementary monotonicity theorems. Not all monotonicity theorems can be approached with this minimum of structure that we have presented so far. We can however give the beginnings of the theory without further complications. By an "elementary' monotonicity theorem we shall mean one that flows without much difficulty directly from the partitioning property. It is remarkable that this one property can be carried so far.

We state the three basic theorems.

THEOREM A. Let B be a family of subsets of $\mathcal{I} \times [a,b]$ with the properties:

pointwise character and the partitioning property.

Then if $\underline{D}_B f(x) \geqq 0$ everywhere on $[a,b]$, f must be nondecreasing on that interval.

THEOREM B. Let B be a family of subsets of $\mathcal{I} \times [a,b]$ with the properties:

pointwise character, finer than the topology, and the partitioning property.

Then if $\underline{D}_B f(x) \geqq 0$ a.e. in $[a,b]$ and $\underline{D}_B f(x) > -\infty$ everywhere on $[a,b]$, f must be nondecreasing on that interval.

THEOREM C. Let B be a family of subsets of $\mathcal{I} \times [a,b]$ with the properties:

pointwise character, filtering down, finer than the topology, Young decomposition, and the partitioning property.

Then if f is a Darboux Baire-1 function on $[a,b]$ for which $\underline{D}_B f(x) = \overline{D}_B f(x)$ holds everywhere in $[a,b]$ with at most countably many exceptions, and $\underline{D}_B f(x) \geqq 0$ holds almost everywhere, f must be continuous and nondecreasing on $[a,b]$.

It is easy to see why the partitioning property should carry a number of monotonicity results. For example if $\underline{D}_B f(x) > 0$ everywhere on $[a,b]$ where B is assumed to have pointwise character and to have the partitioning property then there must exist a $\beta \in B$ for which $f(I) > 0$ for every $(I,x) \in \beta$. But any interval $J \in \mathcal{I}$ permits the existence of a pointed-partition $\pi \subseteq \beta$ of J and this gives

$$f(J) = \Sigma_{(I,x) \in \pi} f(I) > 0 .$$

Consequently f must be strictly increasing on $[a,b]$. Both A and B are proved using this idea. Theorem C can be proved by following very closely the proof of the Goldowski-Tonelli theorem in [9, p. 206] and then applying the reduction theorem of Bruckner referred to in the introduction above.

Each of these theorems applied to the derivation bases for the ordinary, the approximate, the selective, and the qualitative derivatives. The one-sided (Dini) derivation basis and the symmetric derivation bases do not have the partitioning property and a more subtle approach is needed. This leads us to the notion of a variational measure associated with an interval function and a derivation basis.

4. Total variation measures. We wish to present a notion of a measure that somehow reflects the total variation of a function. The most familiar concept that carries this idea is the Lebesgue-Stieltjes measure associated with any function

of bounded variation. As we very definitely do not want to restrict our attention to such a narrow class of functions we will need to depart from the usual constructions.

Various considerations from integration theory and from classical variational ideas lead one to define for any function

$$h : \mathcal{I} \times [a,b] \to R$$

and for any subset β of $\mathcal{I} \times [a,b]$,

$$V(h,\beta) = \sup\{\Sigma_{(I,x) \in \pi} |h(I,x)| : \pi \subseteq \beta\}$$

where the supremum is with regard to all pointed-partitions π contained in β, i.e., finite subsets $\pi = \{(I_1,x_i) : i = 1,\ldots,n\}$ with I_i and I_j nonoverlapping if $i \neq j$. Where there are no such partitions we set $V(h,\emptyset) = 0$. This expresses compactly (much too compactly perhaps) a vast array of technical ideas that are commonly used in analysis. Without presenting the details let us say only that all of the variational computations that appear in Saks [9] can be expressed in this language as well as notions involving upper integrals in various senses. The examples below illustrate.

For any nonempty family B of subsets of $\mathcal{I} \times [a,b]$ and any such interval-point function h we write

$$h_B(X) = \inf_{\beta \in B} V(h,\beta[X]) \quad (X \subseteq [a,b])$$

and refer to h_B as the total variation "measure" associated with h for the derivation basis B.

Example (a). Let U be the uniform derivation basis, example (ii) above, and set $m(I,x) = |I|$. Then $m_U(X)$ is the classical Peano-Jordan content of the set X.

If $h(I,x) = f(x)|I|$ for a bounded function f then $h_U([a,b]) = \overline{\int_a^b} |f(x)|dx$ is the upper Darboux integral of $|f|$.

Example (b). Let D be the derivation basis for the ordinary derivative, example (iii) above, and let $m(I,x) = |I|$ and $h(I,x) = f(x)|I|$ as before. Then $m_D(X)$ is the Lebesgue outer measure of X and $h_D(X) = \overline{\int_X} |f(x)|dx$ is the upper Lebesgue integral of $|f|$ on X.

For a continuous function F on $[a,b]$, considered as defined on $\mathcal{I} \times [a,b]$ by the device $F(I,x) = F(I)$, the measure F_D reflects a number of familiar ideas: F_D is finite if and only if F has bounded variation, and F_D is σ-finite if and only if F is VBG_*; F is AC (respectively ACG_*) if and only if F_D is finite (resp. σ-finite) and absolutely continuous with respect to Lebesgue measure.

These two examples indicate the considerable scope and power of these variational tools. The theory derives mainly from Henstock's attempts to unify much of classical integration theory; see [6] and earlier works of Henstock. The following theorem is essentially his.

THEOREM. Let B be a nonempty family of subsets of $\mathcal{I} \times [a,b]$ and h a real-valued interval-point function. If B has pointwise character then h_B is an outer measure on $[a,b]$. If moreover B is finer than the topology then h_B is a metric outer measure on $[a,b]$ (i.e., all Borel sets are h_B-measurable).

We can now explain how these variational ideas can be brought to bear on the monotonicity problem. Let f be a real-valued function on $[a,b]$, considered as usual as an interval function $f : \mathcal{I} \to R$. Define the negative part of f, f^-, by writing

$$f^-(I) = \max\{0, -f(I)\} .$$

To establish that f is nondecreasing on $[a,b]$ is equivalent, trivially, to showing that the corresponding subadditive interval function f^- vanishes identically. Most frequently this can be proved by showing instead that the associated variational measure f^-_B itself vanishes. Thus we have the ingredients of a monotonicity theorem: (i) a setting in which the vanishing of f^-_B requires the vanishing of f^- and (ii) conditions which ensure the vanishing of f^-_B. It should be clear now why the following definition is needed.

DEFINITION. Let H be a given class of nonnegative subadditive interval functions. Then a family B of subsets of $I \times [a,b]$ is said to be H-complete if $h_B([a,b]) = 0$ and $h \in H$ together imply that $h \equiv 0$.

Any derivation basis that has the partitioning property is evidently H-complete for any choice of H. The most interesting application of this concept is with $H = C$, the class of all continuous subadditive interval functions. The derivation bases for the one-sided derivative and for the symmetric derivatives are C-complete and yet do not have the partitioning property.

5. General monotonicity theorems. In order to capture a great many monotonicity theorems we need a systematic method of handling the exceptional sets that frequently arise. The device suggested by the ideas of the previous section is to introduce appropriate measures and then the exceptional sets more or less take care of themselves. Everything we need is contained in the following two observations, each of which is quite elementary:

(i) Let $\mathcal{B}$ be a family of subsets of $\mathcal{I} \times [a,b]$ that has pointwise character. If $\underline{D}_{\mathcal{B}} f(x) \geq 0$ for every x in a set X then $f^{-}_{\mathcal{B}}(X) = 0$.

(ii) Let $\mathcal{B}$ be a family of subsets of $\mathcal{I} \times [a,b]$ that has pointwise character and is finer than the topology. If $\underline{D}_{\mathcal{B}} f(x) \geq 0$ for almost every x in a set X and $\underline{D}_{\mathcal{B}} f(x) > -\infty$ for every x in X then $f^{-}_{\mathcal{B}}(X) = 0$.

In many settings all that is needed in order to show that f is nondecreasing is that $f^{-}_{\mathcal{B}}([a,b]) = 0$. Thus we have a number of monotonicity results merely in terms of null sets for the measures $f^{-}_{\mathcal{B}}$.

THEOREM A'. Let $\mathcal{B}$ be a family of subsets of $\mathcal{I} \times [a,b]$ that has pointwise character and is $\mathcal{H}$-complete for a family $\mathcal{H}$ of nonnegative subadditive interval functions. Then if $\underline{D}_{\mathcal{B}} f(x) \geq 0$ for $f^{-}_{\mathcal{B}}$-almost every x in $[a,b]$ and $f^{-} \in \mathcal{H}$, f must be nondecreasing on $[a,b]$.

THEOREM B'. Let $\mathcal{B}$ be a family of subsets of $\mathcal{I} \times [a,b]$ that has pointwise character, is finer than the topology, and is $\mathcal{H}$-complete for a family $\mathcal{H}$ of nonnegative subadditive interval functions. Then if $\underline{D}_{\mathcal{B}} f(x) \geq 0$ a.e. in $[a,b]$, $\underline{D}_{\mathcal{B}} f(x) > -\infty$ for $f^{-}_{\mathcal{B}}$-almost every x in $[a,b]$, and $f^{-} \in \mathcal{H}$, f must be nondecreasing on $[a,b]$.

The Goldowski-Tonelli-Zahorski theorem that was given before using the partitioning property can be proved with the weaker one of $\mathcal{C}$-completeness.

THEOREM C'. Let $\mathcal{B}$ be a family of subsets of $\mathcal{I} \times [a,b]$ with the properties:

pointwise character, filtering down, finer than the topology, Young decomposition, and $\mathcal{C}$-complete.

Then if f is a Darboux Baire-1 function on $[a,b]$ for which $\underline{D}_{\mathcal{B}} f(x) = \overline{D}_{\mathcal{B}} f(x)$ holds everywhere with at most countably many exceptions, and $\underline{D}_{\mathcal{B}} f(x) \geqq 0$ a.e. in $[a,b]$, f must be continuous and nondecreasing on $[a,b]$.

These three theorems capture a wide array of monotonicity theorems. In some cases it is easier to prove the monotonicity result directly by invoking properties of the corresponding derivative rather than verifying that the corresponding derivation basis has the listed properties. For example it is proved in [2, p. 189] by quite simple means that the following monotonicity theorem for the upper Dini derivative is true:

if f is continuous, $D^+ f(x) \geqq 0$ a.e. and $D^+ f(x) > -\infty$ everywhere except possibly on a countable set, then f is nondecreasing.

It is nonetheless remarkable that this theorem is a direct corollary of our Theorem B'.

For more difficult monotonicity results it may be better to invoke the general theory, in particular for results that need delicate conditions on the exceptional sets. For example Burkill [4] proved that:

if f is approximately continuous, $\underline{f}'_{ap}(x) \geq 0$ a.e. and $\underline{f}'_{ap}(x) > -\infty$ everywhere then f is nondecreasing.

O'Malley [8] has:

if f is measurable, $\underline{f}'_{ap}(x) \geq 0$ a.e. and $\underline{f}'_{ap}(x) > -\infty$ everywhere then f is nondecreasing.

Then using the general Theorem B' one can allow easily an exceptional countable set in Burkill's result, and make more delicate assertions in O'Malley's result: an exceptional countable set is permitted there provided only that at each of its points f is upper approximately semicontinuous on the left and lower approximately semicontinuous on the right. Such an improvement might be harder to make directly.

We resist the temptation to construct a complete list of monotonicity theorems embraced by these three general theorems.

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