A THEORY OF INTEGRATION

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This paper presents an exposition of the ideas fundamental to a theory of integration which has been investigated by R. Henstock [3], [4] and [5] and later by E. J. McShane [6]. The emphasis of these authors has been on a Riemann-type definition of an integral which possesses Lebesgue-type limit theorems and in particular on the problems of defining such an integral for vector-valued functions.

There is an underlying simplicity in this area which is obscured by a Riemann-oriented approach. We present here, in what seems to be the simplest kind of setting, the basic ideas of that part of the theory which interacts with the measure-theoretic tradition. The generalized versions of Henstock and McShane can then be considered to expand this setting. The paper concludes with a brief application of the theory to the familiar problem of integration in locally compact spaces.

1. Basic theory. Throughout the paper $\mathbf{N}$ will denote the natural numbers, $\mathbf{R}$ the real numbers, $\mathbf{R}_+$ the nonnegative real numbers, $\bar{\mathbf{R}}_+$ the extended nonnegative real numbers and $\bar{\mathbf{R}}_+^T$ the collection of all functions defined on a set T with values in $\bar{\mathbf{R}}_+$.

Let T be a set and $\mathbf{I}$ a collection of pairs (I, x) where $x \in T$ and $I \subseteq T$. A subset $\mathbf{D}$ of $\mathbf{I}$ is said to be *disjointed* if the corresponding collection $\{I\colon (I, x) \in \mathbf{D}\}$ is disjointed; a finite disjointed subset $\mathbf{D}$ of $\mathbf{I}$ is called a *division* and we write $\sigma(\mathbf{D}) = \bigcup \{I : (I, x) \in \mathbf{D}\}$ and call such sets $\sigma(\mathbf{D})$ *elementary sets.* The family of all elementary sets is denoted $\mathfrak{E}$.

If $X \subseteq T$, $\mathbf{S} \subseteq \mathbf{I}$ and $\mathfrak{A}$ is a family of subsets of $\mathbf{I}$, we define the following special sets.

$$\mathbf{S}(X) = \{(I, x) \in \mathbf{S} : I \subseteq X\}$$

$$\mathbf{S}[X] = \{(I, x) \in \mathbf{S} : x \in X\}$$

$$\mathfrak{A}(X) = \{\mathbf{S}(X) : \mathbf{S} \in \mathfrak{A}\}$$

$$\mathfrak{A}[X] = \{\mathbf{S}[X] : \mathbf{S} \in \mathfrak{A}\}$$

DEFINITION 1. The ordered triple $(T, \mathfrak{A}, \mathbf{I})$ is said to be a *division system* if $\mathfrak{A}$ is a collection of subsets of $\mathbf{I}$ directed downwards by set inclusion, i.e., if $\mathbf{S}_1$, $\mathbf{S}_2 \in \mathfrak{A}$, then there is an $\mathbf{S} \in \mathfrak{A}$ such that $\mathbf{S} \subseteq \mathbf{S}_1 \cap \mathbf{S}_2$.

DEFINITION 2. A division system $(T, \mathfrak{A}, \mathbf{I})$ is said to be *fully decomposable* (respectively *decomposable*) if for every family (respectively countable family)

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$\{X_i : i \in I\}$ of disjoint subsets of T and every family $\{\mathbf{S}_i : i \in I\} \subseteq \mathfrak{A}$ there is an $\mathbf{S} \in \mathfrak{A}$ such that $\mathbf{S}[X_i] \subseteq \mathbf{S}_i[X_i]$, $i \in I$. For a function $\mu : \mathbf{I} \to \mathbf{R}$ we define the *variation* of μ with respect to a subset $\mathbf{S}$ of $\mathbf{I}$ as

$$V(\mu, \mathbf{S}) = \sup (\mathbf{D})\sum |\mu(I, x)|$$

where the supremum is with regard to all divisions $\mathbf{D}$, $\mathbf{D} \subseteq \mathbf{S}$, and $(\mathbf{D}) \sum$ denotes a summation over all $(I, x) \in \mathbf{D}$, an empty sum by convention being zero.

If $\mathfrak{A}$ is a family of subsets of $\mathbf{I}$, we define also

$$V(\mu, \mathfrak{A}) = \inf \{V(\mu, \mathbf{S}) : \mathbf{S} \in \mathfrak{A}\}.$$

THEOREM 1. *Let $(T, \mathfrak{A}, \mathbf{I})$ be a decomposable division system, let μ be a real-valued function on $\mathbf{I}$ and denote $\mu^*(X) = V(\mu, \mathfrak{A}[X])$, $X \subseteq T$. Then μ^* is a measure on T.*

Proof. Clearly $\mu^*(\emptyset) = 0$ so that we need only prove that $\mu^*(X) \leq \sum_{j=1}^{\infty} \mu^*(X_j)$ for any sequence $\{X_j\}$ of subsets of T with $X \subseteq \bigcup_{j=1}^{\infty} X_j$. For $\epsilon > 0$ and $j \in \mathbf{N}$ choose $\mathbf{S}_j \in \mathfrak{A}$ such that $V(\mu, \mathbf{S}_j[X_j]) \leq \mu^*(X_j) + \epsilon/2^j$. Since $(T, \mathfrak{A}, \mathbf{I})$ is decomposable there is an $\mathbf{S} \in \mathfrak{A}$ such that $\mathbf{S}[Y_j] \subseteq \mathbf{S}_j[Y_j]$ where $Y_j = X_j \setminus (\bigcup_{k<j} X_k)$, $j \in \mathbf{N}$. Then if $\mathbf{D} \subseteq \mathbf{S}[X]$ is a division, write $\mathbf{D}_j = \mathbf{D}[Y_j]$ and observe that

$$(\mathbf{D}) \sum |\mu(I, x)| = \sum_{j=1}^{\infty} (\mathbf{D}_j) \sum |\mu(I, x)|$$
$$\leq \sum_{j=1}^{\infty} V(\mu, \mathbf{S}_j[X_j]) \leq \sum_{j=1}^{\infty} \mu^*(X_j) + \epsilon.$$

If follows that $\mu^*(X) \leq V(\mu, \mathbf{S}[X]) \leq \sum_{j=1}^{\infty} \mu^*(X_j) + \epsilon$ and as $\epsilon > 0$ is arbitrary the assertion follows.

This method of constructing measures is due to Henstock and does not appear to be widely known. The following regularity property and its role in obtaining a monotone convergence property for measures is also essentially due to Henstock.

DEFINITION 3. A function $\mu : \mathbf{I} \to \mathbf{R}$ is said to be *regular* on $(T, \mathfrak{A}, \mathbf{I})$ provided

$$V(\mu, \mathfrak{A}[X]) = V(\mu, \mathfrak{A}[X](E)) + V(\mu, \mathfrak{A}[X](\setminus E))$$

for every $X \subseteq T$ and every elementary set E. (Here $\setminus E$ denotes $T \setminus E$.)

If χ_X denotes the characteristic function of the set X and if μ is regular, then it should be noted that the function $(I, x) \to \chi_X(x)\mu(I, x)$, denoted $\chi_X\mu$, is also regular.

LEMMA 1. *Let $\mu : \mathbf{I} \to \mathbf{R}$ be regular on a division system $(T, \mathfrak{A}, \mathbf{I})$. If*

$$V(\mu, \mathbf{S}) \leq V(\mu, \mathfrak{A}) + \epsilon < +\infty$$

for some $\mathbf{S} \in \mathfrak{A}$ and $\epsilon > 0$, then

$$V(\mu, \mathbf{S}(E)) \leq V(\mu, \mathfrak{A}(E)) + \epsilon$$

for every elementary set E.

Proof. [4; 231] Clearly

$$V(\mu, \mathbf{S}(E)) \leq V(\mu, \mathbf{S}) - V(\mu, \mathbf{S}(\setminus E))$$
$$\leq V(\mu, \mathfrak{A}) + \epsilon - V(\mu, \mathfrak{A}(\setminus E))$$
$$\leq V(\mu, \mathfrak{A}(E)) + \epsilon$$

where this last inequality results from the assumed regularity of μ.

COROLLARY. *Let $\mu : \mathbf{I} \to \mathbf{R}$ be regular on a division system $(T, \mathfrak{A}, \mathbf{I})$ with $V(\mu, \mathfrak{A}) < +\infty$. Then $V(\mu, \mathfrak{A}) = \sup_{E \varepsilon \mathfrak{E}} V(\mu, \mathfrak{A}(E))$.*

Proof. Let $\epsilon > 0$ and choose $\mathbf{S} \varepsilon \mathfrak{A}$ so that $V(\mu, \mathbf{S}) \leq V(\mu, \mathfrak{A}) + \epsilon$ and choose a division $\mathbf{D} \subseteq \mathbf{S}$ so that $(\mathbf{D}) \sum |\mu(I, x)| \geq V(\mu, \mathbf{S}) - \epsilon$. Then if $E = \sigma(\mathbf{D})$, we have using the lemma

$$V(\mu, \mathfrak{A}(E)) + \epsilon \geq V(\mu, \mathbf{S}(E)) \geq (\mathbf{D}) \sum |\mu(I, x)|$$
$$\geq V(\mu, \mathbf{S}) - \epsilon$$
$$\geq V(\mu, \mathfrak{A}) - \epsilon$$

and the corollary follows easily.

THEOREM 2. *Let μ be a regular on a decomposable division system $(T, \mathfrak{A}, \mathbf{I})$ and write $\mu^*(X) = V(\mu, \mathfrak{A}[X])$, $X \subseteq T$. Then for every increasing sequence of sets $X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots$*

$$\mu^*\left(\bigcup_{k=1}^{\infty} X_k\right) = \lim_{k\to\infty} \mu^*(X_k).$$

Proof. [4; 231] Define $\mu_k = \chi_{X_k}\mu$; we prove the theorem on the more general assumption that each μ_k is regular, a result which will be used in the proof of Theorem 4 later.

For $\epsilon > 0$ and $k \varepsilon \mathbf{N}$ choose $\mathbf{S}_k \varepsilon \mathfrak{A}$ so that

$$V(\mu_k, \mathbf{S}_k) = V(\mu, \mathbf{S}_k[X_k])$$
$$\leq \mu^*(X_k) + \epsilon/2^k$$
$$= V(\mu_k, \mathfrak{A}) + \epsilon/2^k.$$

Since $(T, \mathfrak{A}, \mathbf{I})$ is decomposable there exists an $\mathbf{S} \varepsilon \mathfrak{A}$ such that for each k

$$\mathbf{S}[X_k \setminus X_{k-1}] \subseteq \mathbf{S}_k[X_k \setminus X_{k-1}]$$

where we set $X_0 = \emptyset$.

Let $\mathbf{D} \subseteq \mathbf{S}[\bigcup_{k=1}^{\infty} X_k]$ be an arbitrary division and set $\mathbf{D}_k = \mathbf{D}[X_k \setminus X_{k-1}]$ and $E_k = \sigma(\mathbf{D}_k)$. Let m be the least integer for which $\mathbf{D}_k = \emptyset$, $k > m$.

Then using Lemma 1 applied to each μ_k

$$\begin{aligned}
(\mathbf{D}) \sum |\mu(I, x)| &= \sum_{k=1}^{m} (\mathbf{D}_k) \sum |\mu(I, x)| \\
&\leq \sum_{k=1}^{m} V(\mu_k, \mathbf{S}_k(E_k)) \\
&\leq \sum_{k=1}^{m} V(\mu_k, \mathfrak{A}(E_k)) + \epsilon \\
&\leq \sum_{k=1}^{m} V(\mu_m, \mathfrak{A}(E_k)) + \epsilon \\
&\leq V(\mu_m, \mathfrak{A}) + \epsilon \\
&\leq \mu^*(X_m) + \epsilon.
\end{aligned}$$

Hence

$$\mu^*\left(\bigcup_{k=1}^{\infty} X_k\right) \leq V\left(\mu, \mathbf{S}\left[\bigcup_{k=1}^{\infty} X_k\right]\right) \leq \sup_k \mu^*(X_k) + \epsilon.$$

Since the inequality $\mu^*(X_k) \leq \mu^*(\bigcup_{i=1}^{\infty} X_i)$, $k \in \mathbf{N}$, also holds, the conclusion of the theorem follows on letting $\epsilon \to 0$.

THEOREM 3. *Let μ be regular on a decomposable division system $(T, \mathfrak{A}, \mathbf{I})$ and suppose an elementary set E satisfies*

$$V(\mu, \mathfrak{A}[E](\setminus E)) = V(\mu, \mathfrak{A}[\setminus E](E)) = 0.$$

Then E is μ^*-measurable and $\mu^*(E) = V(\mu, \mathfrak{A}(E))$.

Proof. If μ is regular, then $V(\mu, \mathfrak{A}[E]) = V(\mu, \mathfrak{A}[E](E)) + V(\mu, \mathfrak{A}[E](\setminus E))$. But we also have $V(\mu, \mathfrak{A}[E](E)) \leq V(\mu, \mathfrak{A}(E)) \leq V(\mu, \mathfrak{A}[E](E)) + V(\mu, \mathfrak{A}[\setminus E](E))$. Combining these statements with the hypothesis of the theorem yields $\mu^*(E) = V(\mu, \mathfrak{A}(E))$. Similarly $\mu^*(\setminus E) = V(\mu, \mathfrak{A}(\setminus E))$.

Let $X \subseteq T$ and write $\mu_0 = \chi_X \mu$; then μ_0 is also regular and the above results yield

$$\begin{aligned}
\mu^*(X) = V(\mu_0, \mathfrak{A}) &= V(\mu_0, \mathfrak{A}(E)) + V(\mu_0, \mathfrak{A}(\setminus E)) \\
&= \mu_0^*(E) + \mu_0^*(\setminus E) \\
&= \mu^*(X \cap E) + \mu^*(X \setminus E)
\end{aligned}$$

so that E is μ^*-measurable as required.

Let f be a real-valued function on T and suppose μ is a real-valued function on $\mathbf{I}$; we denote the function $(I, x) \to f(x)\mu(I, x)$ by $f\mu$ and consider the mapping $f \to V(f\mu, \mathfrak{A})$. This behaves much like an upper integral and we formalize this in a definition.

DEFINITION 4. Let $\mu : \mathbf{I} \to \mathbf{R}$ and let $(T, \mathfrak{A}, \mathbf{I})$ be a division system; then for every function $f \in \bar{\mathbf{R}}_+^T$ we write $N_\mu(f) = V(f\mu, \mathfrak{A})$ if f is finite everywhere and otherwise $N_\mu(f) = \sup_m V(f^{(m)}\mu, \mathfrak{A})$, where $f^{(m)}(x) = f(x)$ if $f(x) < +\infty$ and $f^{(m)}(x) = m$ if $f(x) = +\infty$.

The following properties for N_μ can be easily obtained.

(1) If $0 \leq f(x) \leq g(x)$, $x \in T$, then $N_\mu(f) \leq N_\mu(g)$.

(2) If $c \geq 0$ and $f \in \bar{\mathbf{R}}_+^T$, then $N_\mu(cf) = cN_\mu(f)$.

(3) If $f, g \in \bar{\mathbf{R}}_+^T$, then $N_\mu(f + g) \leq N_\mu(f) + N_\mu(g)$.

A function $N_\mu : \bar{\mathbf{R}}_+^T \to \bar{\mathbf{R}}_+$ which in addition to satisfying (1), (2) and (3) above also satisfies (4) below is said to be an *upper integral* on T.

(4) If $\{f_n\}$ is an increasing sequence in $\bar{\mathbf{R}}_+^T$ and $f(x) = \sup_n f_n(x)$, $x \in T$, then $N_\mu(f) = \sup_n N_\mu(f_n)$.

THEOREM 4. *Let $(T, \mathfrak{A}, \mathbf{I})$ be a decomposable division system and suppose $\mu : \mathbf{I} \to \mathbf{R}$ is such that $f\mu$ is regular for every bounded function on T. Then N_μ is an upper integral on T.*

Proof. We need prove only the statement (4) above. Let $\{f_n\}$ and f be as in that statement and define $f^{(m)}$ as in Definition 4.

For each $x \in T$ and $m \in \mathbf{N}$ let $n(x, m)$ denote the least integer for which

$$f_{n(x,m)}(x) \geq \left(\frac{m-1}{m}\right) f^{(m)}(x)$$

and define

$$X_{nm} = \{x \in T : n(x, m) \leq n, f(x) \leq n\}.$$

Then

$$\left(\frac{m-1}{m}\right) V(f^{(m)}\mu, \mathfrak{A}[X_{nm}]) \leq N_\mu(f_n) \leq \sup_k N_\mu(f_k).$$

But $\chi_{X_{nm}} f^{(m)}$ is bounded so that by the hypothesis of the theorem $\chi_{X_{nm}} f^{(m)}\mu$ is regular. Applying then Theorem 2 (see the first paragraph of the proof) and letting $n \to \infty$ we obtain

$$\left(\frac{m-1}{m}\right) V(f^{(m)}\mu, \mathfrak{A}) \leq \sup_k N_\mu(f_k).$$

Now letting $m \to \infty$ proves that

$$N_\mu(f) \leq \sup_k N_\mu(f_k)$$

which, since the opposite inequality is obvious, completes the proof.

An upper integral as defined above gives rise to a theory of the integral in a standard manner. We summarize these methods briefly.

DEFINITION 5. Let $\mathfrak{F}(T, N_\mu)$ denote the linear space of all functions $f : T \to \mathbf{R}$ such that $N_\mu(|f|) < +\infty$ and consider $\mathfrak{F}(T, N_\mu)$ endowed with the topology defined by the seminorm N_μ.

Let $\mathfrak{N}$ be a subspace of $\mathfrak{F}(T, N_\mu)$ such that (i) if $\varphi:\mathbf{R} \rightarrow \mathbf{R}$ is continuous with $\varphi(0) = 0$ and if $f \in \mathfrak{N}$, then $\varphi \circ f \in \mathfrak{N}$ and (ii) $N_\mu(f + g) = N_\mu(f) + N_\mu(g)$ for nonnegative f, g in $\mathfrak{N}$. Then there is a unique continuous linear functional $f \rightarrow \int f\, d\mu$ called *the integral* (with respect to N_μ and $\mathfrak{N}$) defined on $\mathfrak{L}$, the closure of $\mathfrak{N}$ in $\mathfrak{F}(T, N_\mu)$, which is an extension of $f \rightarrow N_\mu(f)$ on $\{f \in \mathfrak{N} : f \geq 0\}$. (Note that for $f \in \mathfrak{L}$, $f \geq 0$, $\int f\, d\mu = N_\mu(f)$.)

The choice of a suitable $\mathfrak{N}$ can depend on the context but in the present situation with no additional structure present one can take $\mathfrak{N}$ as the space of all μ^*-measurable simple functions and the resulting integral coincides with the measure theoretic integral.

2. Integration in locally compact spaces. Throughout this section T will be a locally compact Hausdorff space, $\mathfrak{K}$ the family of all compact subsets of T and $\mathbf{I}$ the collection of all pairs (I, x), $I \in \mathfrak{K}$, $x \in I$. If N is an arbitrary function on T such that each $N(x)$ is a neighbourhood of the point x, we define the subset $\mathbf{S}_N \subseteq \mathbf{I}$ by $\mathbf{S}_N = \{(I, x) : x \in I \subseteq N(x), I \in \mathfrak{K}\}$. If $\mathfrak{A}$ denotes the collection of all such $\mathbf{S}_N$, then it is easy to see that $(T, \mathfrak{A}, \mathbf{I})$ is a fully decomposable division system. Note that here the family $\mathfrak{E}$ of all elementary sets coincides with $\mathfrak{K}$.

DEFINITION 6. [2; 231] A function $\lambda : \mathfrak{K} \rightarrow \mathbf{R}_+$ is said to be a *content* if (i) $K_1 \subseteq K_2$ entails $\lambda(K_1) \leq \lambda(K_2)$, (ii) $K_1 \cap K_2 = \emptyset$ entails $\lambda(K_1 \cup K_2) = \lambda(K_1) + \lambda(K_2)$ and (iii) $\lambda(K_1 \cup K_2) \leq \lambda(K_1) + \lambda(K_2)$ where K_1, $K_2 \in \mathfrak{K}$.
If in addition for every $K \in \mathfrak{K}$ and every $\epsilon > 0$ there is an open set G with $K \subseteq G$ such that $|\lambda(C) - \lambda(C \cap K)| < \epsilon$ whenever $C \subseteq G$, $C \in \mathfrak{K}$, then λ is said to be a *regular content.*

THEOREM 5. *Let λ be a regular content and suppose $f : T \rightarrow \mathbf{R}$ is bounded; then the function $f\lambda$: $(I, x) \rightarrow f(x)\lambda(I)$, $(I, x) \in \mathbf{I}$, is regular.*

Proof. If $\epsilon > 0$ and $K \in \mathfrak{K}$, there is an open set $U \supseteq K$ such that $|\lambda(C) - \lambda(C \cap K)| < \epsilon/M$ for every $C \subseteq U$, $C \in \mathfrak{K}$, where $M > \sup_{x \in T} |f(x)|$. Choose $\mathbf{S} \in \mathfrak{A}$ such that $\mathbf{S}[U] \subseteq \mathbf{S}(U)$ and $\mathbf{S}[\diagdown K] \subseteq \mathbf{S}(\diagdown K)$ (This is possible by the definition of $\mathfrak{A}$.).
If $\mathbf{D} \subseteq \mathbf{S}$ is a division, then using the fact that $\mathbf{D}[\diagdown K] = \mathbf{D}(\diagdown K)$

$$\begin{aligned}(\mathbf{D}) \sum |f(x)|\, \lambda(I) = (\mathbf{D}[K]) \sum \{|f(x)|\, \lambda(I \cap K) + |f(x)|\, (\lambda(I) \\ - \lambda(I \cap K))\} + \mathbf{D}(\diagdown K) \sum |f(x)|\, \lambda(I) \\ \leq V(f\lambda, \mathbf{S}(K)) + \epsilon + V(f\lambda, \mathbf{S}(\diagdown K)).\end{aligned}$$

From this it can be deduced that $V(f\lambda, \mathfrak{A}) = V(f\lambda, \mathfrak{A}(K)) + V(f\lambda, \mathfrak{A}(\diagdown K))$. Since this is true for all $K \in \mathfrak{K}$ and for all bounded functions f it follows that $f\lambda$ is regular as stated in the theorem.

In particular we conclude from this theorem that the upper integral N_λ generated by a regular content λ satisfies Theorem 4 and this provides a definition for the integral in locally compact spaces relative to an arbitrary regular content defined on the compact subsets of T.

THEOREM 6. *Let λ be a regular content and write $\mu(I, x) = \lambda(I)$, $(I, x) \in \mathbf{I}$.*

Then μ is regular on $(T, \mathfrak{A}, \mathbf{I})$, every Borel set is μ^*-measurable, and the restriction of μ^* to the Borel sets is a regular Borel measure which coincides on $\mathfrak{K}$ with λ.

Proof. It follows from the previous theorem that the function μ is regular on $(T, \mathfrak{A}, \mathbf{I})$. It is evident from the manner in which $\mathfrak{A}$ was constructed that for every $K \in \mathfrak{K}$

$$V(\mu, \mathfrak{A}[K](\setminus K)) = V(\mu, \mathfrak{A}[\setminus K](K)) = 0$$

so that by Theorem 3 each set in $\mathfrak{K}$ is μ^*-measurable and hence so is every Borel set.

In order to prove that the restriction of μ^* to the Borel sets is a regular Borel measure it suffices [2; 228] to show that $\mu^*(K) = \inf \mu^*(U)$, $K \subseteq U$, U open, for each $K \in \mathfrak{K}$.

Let $\epsilon > 0$ and $K \in \mathfrak{K}$ and choose U open, $U \supseteq K$, such that if $C \in \mathfrak{K}$ and $C \subseteq U$, then $|\lambda(C) - \lambda(C \cap K)| < \epsilon$. Let $\mathbf{S} \in \mathfrak{A}$ be chosen arbitrarily subject to the conditions that $\mathbf{S}[U] \subseteq \mathbf{S}(U)$ and $\mathbf{S}[\setminus K] \subseteq \mathbf{S}(\setminus K)$. Then we have by using a similar argument to that used in the previous theorem

$$V(\mu, \mathbf{S}[U]) \leq V(\mu, \mathbf{S}(U)) \leq V(\mu, \mathbf{S}[K]) + \epsilon$$

from which it can be shown that $\mu^*(U) \leq \mu^*(K) + \epsilon$ and hence that $\mu^*(K) = \inf \mu^*(U)$, $U \supseteq K$, U open, as required.

Finally it remains to show that $\lambda(K) = \mu^*(K)$, $K \in \mathfrak{K}$. Clearly $\lambda(K) \geq \mu^*(K)$, $K \in \mathfrak{K}$; to show the opposite inequality we state a lemma.

LEMMA. *Let λ_0 be a regular Borel measure on T with $K \in \mathfrak{K}$ and $\mathbf{S} \in \mathfrak{A}$. Then there is a denumerable disjointed set $\mathbf{D}_0 \subseteq \mathbf{S}(K)$ such that $\lambda_0(N) = 0$ where $N = K \setminus \bigcup \{I : (I, x) \in \mathbf{D}_0\}$.*

Proof. The arguments of [1; 191, Proposition 14] can be modified to supply a proof. We omit the details.

Continuing the proof of the theorem, let $K \in \mathfrak{K}$ and $\mathbf{S} \in \mathfrak{A}$, let λ_0 be the regular Borel measure which extends λ [2; 237] and let $\mathbf{D}_0$ be the subset of $\mathbf{S}(K)$ whose existence the lemma asserts. Then

$$\lambda(K) = \lambda_0(K) = \lambda_0(\bigcup \{I : (I, x) \in \mathbf{D}_0\}) = \sum \{\lambda(I) : (I, x) \in \mathbf{D}_0\} \leq V(\mu, \mathbf{S}(K)).$$

Thus $\lambda(K) \leq V(\mu, \mathfrak{A}(K))$. As this latter equals $\mu^*(K)$, by Theorem 3, the proof is completed.

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