ON McSHANE'S VECTOR-VALUED INTEGRAL

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E. J. McShane has recently given an abstract version [3] of a generalization of the classical Riemann integral due (independently) to J. Kurzweil and R. Henstock. Simultaneously, Henstock's own abstract version [2] has appeared. Although there is much in common between these two treatments there are two distinct ideas which do not overlap in their presentations, namely, Henstock's notion of "variation" and McShane's concept of "absolute integrability".

In this paper we show how McShane's "absolute integrability" can be studied in a context derived mainly from Henstock's version of the abstract theory. In particular the underlying measure theory is shown to emerge quite naturally from this point of view.

Notation. The notation used is similar to that of the preceding paper except that the system $(T, \mathfrak{A}, \mathbf{I})$ will arise out of some algebraic structures and the resulting theory will then have more algebraic properties. Let $\mathbf{B}$ be a nonempty class of subsets of a set T. We require the following.

$\mathbf{B}$ is called a *clan* if $A \setminus B \in \mathbf{B}$ and $A \cup B \in \mathbf{B}$ for all $A, B \in \mathbf{B}$.

$\mathbf{B}$ is called a *semiclan* if (i) $A \cap B \in \mathbf{B}$ for $A, B \in \mathbf{B}$ and (ii) for every pair (A, B) of sets of $\mathbf{B}$ for which $A \subseteq B$ there exists a finite family $\{C_0, C_1, C_2, \cdots, C_n\}$ of sets of $\mathbf{B}$ such that $A = C_0 \subseteq C_1 \subseteq \cdots \subseteq C_n = B$ and $C_i \setminus C_{i-1} \in \mathbf{B}$ for $i = 1, 2, \cdots, n$.

$\mathbf{B}$ is called a *tribe* if (i) $A \setminus B \in \mathbf{B}$ for $A, B \in \mathbf{B}$ and (ii) $\bigcup_{i=1}^{\infty} A_i \in \mathbf{B}$ for every sequence $\{A_i\}$ of sets in $\mathbf{B}$.

$\mathbf{B}$ is called a *semitribe* if (i) $A \setminus B \in \mathbf{B}$ for $A, B \in \mathbf{B}$, (ii) $A \cup B \in \mathbf{B}$ for $A, B \in \mathbf{B}$ and (iii) $\bigcap_{i=1}^{\infty} A_i \in \mathbf{B}$ for every sequence $\{A_i\}$ of sets in $\mathbf{B}$.

1. Partitioning systems. Let T be a set, $\mathfrak{B}$ a semiclan [1; 7] (semiring) of subsets of T and $\mathfrak{C}$ the clan (ring) generated by $\mathfrak{B}$. The sets in $\mathfrak{C}$ will be called elementary sets. We write $\mathbf{I} = \mathfrak{B} \times T$. A finite subset $\mathbf{D}$ of $\mathbf{I}$ is called a *partition* if the sets $\{I : (I, x) \in \mathbf{D}\}$ are disjointed. We write then $\sigma(\mathbf{D}) = \bigcup \{I : (I, x) \in \mathbf{D}\}$ and we call $\mathbf{D}$ a *partition of the elementary set* $\sigma(\mathbf{D})$. A subset $\mathbf{S}$ of $\mathbf{I}$ is said to partition E if $\mathbf{S}$ contains a partition of E.

For any subset $\mathbf{S}$ of $\mathbf{I}$ and for any family $\mathfrak{A}$ of subsets of $\mathbf{I}$ we denote

$$\mathbf{S}[X] = \{(I, x) \in \mathbf{S} : x \in X\}$$
$$\mathbf{S}(X) = \{(I, x) \in \mathbf{S} : I \subseteq X\}$$
$$\mathfrak{A}[X] = \{\mathbf{S}[X] : \mathbf{S} \in \mathfrak{A}\}$$
$$\mathfrak{A}(X) = \{\mathbf{S}(X) : \mathbf{S} \in \mathfrak{A}\}.$$

Received March 10, 1972. Revision received June 8, 1972.

DEFINITION 1. An ordered triple $(T, \mathfrak{A}, \mathbf{I})$ is said to be a *partitioning system* if $\mathfrak{A}$ is a collection of subsets of $\mathbf{I}$ satisfying the following conditions.
(1.1) Every $\mathbf{S} \in \mathfrak{A}$ partitions every elementary set.
(1.2) If (I_1, x_1) and (I_2, x_2) belong to an $\mathbf{S} \in \mathfrak{A}$, then so does $(I_1 \cap I_2, x_1)$.
(1.3) $\mathfrak{A}$ is directed downwards by set inclusion.

This is considerably more restrictive than the concept of a division system or of a division space [2], [4] and [5] but presents the most attractive setting for the results of this paper. Many of the results stated are true in more general circumstances but McShane's concept of absolute integrability is best exploited under these assumptions.

DEFINITION 2. A partitioning system $(T, \mathfrak{A}, \mathbf{I})$ is said to be *fully decomposable* (respectively *decomposable*) if for every family (respectively countable family) $\{X_i : i \in I\}$ of disjoint subsets of T and for every family $\{\mathbf{S}_i : i \in I\} \subseteq \mathfrak{A}$ there is an $\mathbf{S} \in \mathfrak{A}$ such that $\mathbf{S}[X_i] \subseteq \mathbf{S}_i[X_i]$ for each $i \in I$.

In [3] fully decomposable systems are said to be of *pointwise character*; the definition and terminology here are due to Henstock [2; 512].

Let μ be a function defined on $\mathbf{I}$ and with values in a normed linear space $\mathbf{E}$. For any subset $\mathbf{S}$ of $\mathbf{I}$ and for any family $\mathfrak{A}$ of subsets of $\mathbf{I}$ we define the *variation* of μ with respect to $\mathbf{S}$ and $\mathfrak{A}$ as

$$V(\mu, \mathbf{S}) = \sup_{\mathbf{D} \subseteq \mathbf{S}} (\mathbf{D}) \sum \|\mu(I, x)\|, \tag{1.4}$$

where $\mathbf{D}$ denotes an arbitrary partition and the sum $(\mathbf{D}) \sum$ is with regard to all $(I, x) \in \mathbf{D}$, an empty sum being replaced by zero, and as

$$V(\mu, \mathfrak{A}) = \inf_{\mathbf{S} \in \mathfrak{A}} V(\mu, \mathbf{S}). \tag{1.5}$$

Henstock [2] has observed that for decomposable partitioning systems $(T, \mathfrak{A}, \mathbf{I})$ the function $X \rightarrow \mu^*(X) = V(\mu, \mathfrak{A}[X])$ defined for all subsets of T is an (outer) measure on T. This is true for any function $\mu: \mathbf{I} \rightarrow \mathbf{E}$. However more specific results are obtainable for those μ that are *regular* [4], i.e., for those which

$$V(\mu, \mathfrak{A}[X]) = V(\mu, \mathfrak{A}[X](E)) + V(\mu, \mathfrak{A}[X](\setminus E))$$

is true for all $X \subseteq T$, $E \in \mathfrak{E}$. (Here we use $\setminus E$ to denote $T \setminus E$.)

If we agree to call a function $\mu: \mathbf{I} \rightarrow \mathbf{E}$ additive (subadditive) and if the function $I \rightarrow \mu(I, x)$, $(I, x) \in \mathbf{I}$, is additive (subadditive) for each fixed $x \in T$, then [5] proves that for partitioning systems $(T, \mathfrak{A}, \mathbf{I})$ every additive (or nonnegative and subadditive) function μ is regular. This result accounts for quite a large part of the theory. We summarize below. Each of these results is true in more general circumstances. (By $\mathbf{R}_+^T$ below we mean the set of all nonnegative real-valued functions on T and $f\mu$ denotes the function $(I, x) \rightarrow f(x)\mu(I, x)$, $(I, x) \in \mathbf{I}$.)

THEOREM 1. *Let $(T, \mathfrak{A}, \mathbf{I})$ be a decomposable partitioning system and let $\mu: \mathbf{I} \rightarrow \mathbf{E}$ be additive (or nonnegative and subadditive). For $X \subseteq T$ and $f \in \mathbf{R}_+^T$ we*

define $\mu^*(X) = V(\mu, \mathfrak{A}[X])$ *and* $N_\mu(f) = V(f\mu, \mathfrak{A})$. *Then the following hold.*
(1) μ^* *is a measure on* T.
(2) *For any increasing sequence of sets* $X_1, X_2, \cdots$ *in* T, $\mu^*(\bigcup_{k=1}^\infty X_k) = \lim_{n\to\infty} \mu^*(X_n)$.
(3) *If* $N_\mu(f) < +\infty$, *then* $N_\mu(f) = \sup_{E \varepsilon \mathfrak{E}} V(f\mu, \mathfrak{A}(E))$.
(4) *The function* $f \to N_\mu(f)$ *is an upper integral on* $\mathbf{R}_+^T$.

Proof. For a proof see [4].

Definition 3. The *integral* of a function $\mu:\mathbf{I} \to \mathbf{E}$ on an elementary set E with respect to a partitioning system $(T, \mathfrak{A}, \mathbf{I})$ is defined as follows. Let $\mathfrak{F}(\mu, \mathbf{S}, E)$ denote the set of all elements in $\mathbf{E}$ which can be written as $(\mathbf{D}) \sum \mu(I, x)$ where $\mathbf{D} \subseteq \mathbf{S}$ is an arbitrary partition of E. Clearly $\{\mathfrak{F}(\mu, \mathbf{S}, E):\mathbf{S} \varepsilon \mathfrak{A}\}$ is a base for a filter in $\mathbf{E}$ whose limit, if it exists, is denoted by $\int_{(E)}\mu$.

If $\int_{(E)}\mu$ exists for every $E \varepsilon \mathfrak{E}$ and $\lim_{E \varepsilon \mathfrak{E}} \int_{(E)}\mu$ exists (in the sense of the net on $\mathfrak{E}$ directed upward by set inclusion), then μ is said to be *integrable* and we write $\int_T \mu = \lim_{E \varepsilon \mathfrak{E}} \int_{(E)} \mu$.

In particular if T itself is an elementary set, then certainly $\int_T\mu = \int_{(T)}\mu$. We state the results we require in a lemma.

Lemma 1. *Let* $(T, \mathfrak{A}, \mathbf{I})$ *be a partitioning system and suppose* $\mu:\mathbf{I} \to \mathbf{E}$ *is additive. Then the following hold.*
(1) *If* $\int_{(E)}\mu$ *exists for all* $E \varepsilon \mathfrak{E}$, *then* $E \to \int_{(E)}\mu$ *is finitely additive on* $\mathfrak{E}$ *and* $\|\int_{(E)}\mu\| \leq V(\mu, \mathfrak{A})$.
(2) *If* $\int_{(E)}\mu$ *exists for all* $E \varepsilon \mathfrak{E}$, $\mathbf{E}$ *is complete and* $V(\mu, \mathfrak{A}) < +\infty$, *then* $\int_T\mu$ *exists and* $\|\int_T\mu\| \leq V(\mu, \mathfrak{A})$.
(3) *If* μ_1 *and* μ_2 *are integrable, then so is* $\mu_1 + \mu_2$ *and* $\int_T(\mu_1 + \mu_2) = \int_T\mu_1 + \int_T\mu_2$.

Proof. (See [2] and [3].) The proof offers no difficulty and is omitted. Assertion (2) simply uses the regularity of μ to show that the net $\{\int_{(E)}\mu : E \varepsilon \mathfrak{E}\}$ is Cauchy in $\mathbf{E}$.

2. Absolute integration theory. Throughout the remainder of the paper $(T, \mathfrak{A}, \mathbf{I})$ will denote a fixed but arbitrary partitioning system. The exposition is intended to establish the relationship the present theory has with the traditional measure-theoretic treatment [1] of the Lebesgue integral for vector-valued functions.

Definition 4. Suppose μ is a function on $\mathbf{I}$ with values in a normed linear space. A set $X \subseteq T$ is said to be μ-*measurable* in $(T, \mathfrak{A}, \mathbf{I})$ if for every $\epsilon > 0$ there is an $\mathbf{S} \varepsilon \mathfrak{A}$ such that

$$(2.1) \qquad (\mathbf{D}_1[X] \times \mathbf{D}_2[\setminus X]) \sum \|\mu(I_1 \cap I_2, x_1)\| \leq \epsilon$$

and

$$(2.2) \qquad (\mathbf{D}_1[X] \times \mathbf{D}_2[\setminus X]) \sum \|\mu(I_1 \cap I_2, x_2)\| \leq \epsilon$$

for all partitions $\mathbf{D}_1$ and $\mathbf{D}_2$ contained in $\mathbf{S}$. (The summation is with regard to all $(I_1, x_1) \in \mathbf{D}_1[X]$ and all $(I_2, x_2) \in \mathbf{D}_2[\setminus X]$.)

In particular if μ is independent of $x \in T$, i.e., if $\mu(I, x) = \mu(I)$ for $(I, x) \in \mathbf{I}$, then (2.1) and (2.2) reduce to the simpler assertion that

$$(\mathbf{D}_1[X] \times \mathbf{D}_2[\setminus X]) \sum \|\mu(I_1 \cap I_2)\| \leq \epsilon.$$

The collection of all μ-measurable sets in $(T, \mathfrak{A}, \mathbf{I})$ is denoted $\mathfrak{M}(\mu)$; clearly $\mathfrak{M}(\mu)$ contains at least the set $\emptyset$ and T. The subfamily of $\mathfrak{M}(\mu)$ consisting of all μ-measurable sets on which μ^* is finite is denoted $\mathfrak{M}_0(\mu)$; clearly $\emptyset \in \mathfrak{M}_0(\mu)$.

LEMMA 2. *Let μ be an additive function on* $\mathbf{I}$ *with values in a normed linear space. Then $\mathfrak{M}(\mu)$ and $\mathfrak{M}_0(\mu)$ are clans; if in addition $(T, \mathfrak{A}, \mathbf{I})$ is decomposable, then $\mathfrak{M}(\mu)$ is a tribe and $\mathfrak{M}_0(\mu)$ is a semitribe.*

Proof. We first prove that $\mathfrak{M}(\mu)$ is a clan; that $\mathfrak{M}_0(\mu)$ is also a clan then follows. If X and Y belong to $\mathfrak{M}(\mu)$ and $\mathbf{D}_1$ and $\mathbf{D}_2$ are any partitions, then

$$\mathbf{D}_1[X \cup Y] \times \mathbf{D}_2[\setminus(X \cup Y)] \subseteq (\mathbf{D}_1[X] \times \mathbf{D}_2[\setminus X]) \cup (\mathbf{D}_1[Y] \times \mathbf{D}_2[\setminus Y])$$

from which, using the additivity of μ, it is easy to show that $X \cup Y \in \mathfrak{M}(\mu)$. By the symmetry of Definition 4, $\mathfrak{M}(\mu)$ is clearly closed under complementation and so it follows that $\mathfrak{M}(\mu)$ is a clan.

Now suppose $(T, \mathfrak{A}, \mathbf{I})$ is decomposable and that $X = \bigcup_{i=1}^{\infty} X_i$, where the $\{X_i\}$ are disjointed and belong to $\mathfrak{M}(\mu)$. Choose an $\mathbf{S}_1 \in \mathfrak{A}$ so that the sums in (2.1) and (2.2) for X_1 are smaller than $\epsilon/2$; proceeding inductively choose $\mathbf{S}_n \in \mathfrak{A}$, $n > 1$, so that the sums in (2.1) and (2.2) for X_n are smaller than $\epsilon/2^{n+1}$ and so that $\mathbf{S}_n \subseteq \mathbf{S}_{n-1}$.

By the decomposability property we select $\mathbf{S} \in \mathfrak{A}$ so that $\mathbf{S}[X_i] \subset \mathbf{S}_i[X_i]$, $i = 1, 2, \cdots$. Then given any pair of divisions $\mathbf{D}_1$ and $\mathbf{D}_2$ contained in $\mathbf{S}$

$$\mathbf{D}_1[X] \times \mathbf{D}_2[\setminus X] \subseteq \bigcup_{n=1}^{\infty} (\mathbf{D}_1[X_n] \times \mathbf{D}_2[\setminus X_n])$$

and an obvious argument yields

$$(\mathbf{D}_1[X] \times \mathbf{D}_2[\setminus X]) \sum \|\mu(I_1 \cap I_2, x_1)\| \leq \epsilon$$

with the corresponding sum in (2.2) also not exceeding ϵ. Thus $X \in \mathfrak{M}(\mu)$ and $\mathfrak{M}(\mu)$ is a tribe; that $\mathfrak{M}_0(\mu)$ is a semitribe is now clear and the lemma is proved.

DEFINITION 5. [3; 15] A function μ on $\mathbf{I}$ with values in a normed linear space is said to be *absolutely integrable* in $(T, \mathfrak{A}, \mathbf{I})$ if for every $\epsilon > 0$ there is an $\mathbf{S} \in \mathfrak{A}$ such that

$$(2.3) \qquad (\mathbf{D}_1 \times \mathbf{D}_2) \sum \|\mu(I_1 \cap I_2, x_1) - \mu(I_1 \cap I_2, x_2)\| \leq \epsilon$$

for every pair of partitions $\mathbf{D}_1$ and $\mathbf{D}_2$ contained in $\mathbf{S}$.

In particular note that any function μ on $\mathbf{I}$ which is independent of $x \in T$ is absolutely integrable.

THEOREM 2. *Let μ be absolutely integrable in $(T, \mathfrak{A}, \mathbf{I})$. A set $X \varepsilon T$ belongs to $\mathfrak{M}(\mu)$ if and only if $\chi_X \mu$ is absolutely integrable; $\mathfrak{M}(\mu)$ contains every set X for which $\mu^*(X) = 0$. If in addition μ is additive and $(T, \mathfrak{A}, \mathbf{I})$ is decomposable, then μ^* is countably additive on $\mathfrak{M}(\mu)$.*

Proof. Suppose firstly that $\chi_X \mu$ is absolutely integrable and let $\mathbf{S} \varepsilon \mathfrak{A}$ be chosen so that

$$(\mathbf{D}_1 \times \mathbf{D}_2) \sum ||\chi_X(x_1)\mu(I_1 \cap I_2, x_1) - \chi_X(x_2)\mu(I_1 \cap I_2, x_2)|| \leq \epsilon$$

for arbitrary $\mathbf{D}_1, \mathbf{D}_2 \subseteq \mathbf{S}$. On replacing $\mathbf{D}_1$ by $\mathbf{D}_1[X]$ and $\mathbf{D}_2$ by $\mathbf{D}_2[\diagdown X]$ we get

$$(\mathbf{D}_1[X] \times \mathbf{D}_2[\diagdown X]) \sum ||\mu(I_1 \cap I_2, x_1)|| \leq \epsilon.$$

A similar argument applied to $\chi_{T \setminus X}\mu = \mu - \chi_X \mu$ (which is also absolutely integrable) yields the statement

$$(\mathbf{D}_1[X] \times \mathbf{D}_2[\diagdown X]) \sum ||\mu(I_1 \cap I_2, x_2)|| \leq \epsilon$$

which together with the above proves that X is μ-measurable.

Conversely suppose that X is μ-measurable and that the sums in (2.1) and (2.2) do not exceed $\epsilon/3$ for partitions $\mathbf{D}_1$ and $\mathbf{D}_2$ contained in an $\mathbf{S} \varepsilon \mathfrak{A}$ and suppose also that the sum in (2.3) does not exceed $\epsilon/3$ (This is possible because μ is absolutely integrable by hypothesis.).

Then for $\mathbf{D}_1, \mathbf{D}_2$ contained in this $\mathbf{S}$

$$\begin{aligned}(\mathbf{D}_1 \times \mathbf{D}_2) \sum ||\chi_X(x_1)\mu(I_1 \cap I_2, x_1) &- \chi_X(x_2)\mu(I_1 \cap I_2, x_2)|| \\ &\leq (\mathbf{D}_1[X] \times \mathbf{D}_2[X]) \sum ||\mu(I_1 \cap I_2, x_1) - \mu(I_1 \cap I_2, x_2)|| \\ &\quad + (\mathbf{D}_1[X] \times \mathbf{D}_2[\diagdown X]) \sum ||\mu(I_1 \cap I_2, x_1)|| \\ &\quad + (\mathbf{D}_2[X] \times \mathbf{D}_1[\diagdown X]) \sum ||\mu(I_1 \cap I_2, x_2)|| \\ &\leq \epsilon\end{aligned}$$

so that $\chi_X \mu$ is absolutely integrable as required.

Now suppose $X \subseteq T$ with $\mu^*(X) = 0$; we will show that $X \varepsilon \mathfrak{M}(\mu)$. Choose $\mathbf{S} \varepsilon \mathfrak{A}$ so that $V(\mu, \mathbf{S}[X]) \leq \epsilon/2$ and so that the sum in (2.3) for μ does not exceed $\epsilon/2$.

Then if $\mathbf{D}_1$ and $\mathbf{D}_2$ are contained in $\mathbf{S}$ so is every $(I_1 \cap I_2, x_1)$ for $(I_1, x_1) \varepsilon \mathbf{D}_1$ and $(I_2, x_2) \varepsilon \mathbf{D}_2$ (Definition 1(2)); hence

$$(\mathbf{D}_1[X] \times \mathbf{D}_2[\diagdown X]) \sum ||\mu(I_1 \cap I_2, x_1)|| \leq V(\mu, \mathbf{S}[X]) \leq \epsilon/2$$

and

$$\begin{aligned}(\mathbf{D}_1[X] \times \mathbf{D}_2[\diagdown X]) &\sum ||\mu(I_1 \cap I_2, x_2)|| \\ &\leq (\mathbf{D}_1[X] \times \mathbf{D}_2[\diagdown X]) \sum ||\mu(I_1 \cap I_2, x_1)|| \\ &\quad + (\mathbf{D}_1[X] \times \mathbf{D}_2[\diagdown X]) \sum ||\mu(I_1 \cap I_2, x_2) - \mu(I_1 \cap I_2, x_1)|| \\ &\leq \epsilon\end{aligned}$$

so that X is μ-measurable as required.

To obtain the final statement of the theorem we observe that μ^* is finitely additive at least on $\mathfrak{M}(\mu)$ (See [5; Theorem 6] which is easily extended to $||\mu||$ subadditive in our context.) and the result on countable additivity follows in a standard manner from Theorem 1(2).

THEOREM 3. *If μ is an absolutely integrable additive function on* $\mathbf{I}$ *with values in a Banach space, then $\int_{(E)}\mu$ exists for every $E \in \mathfrak{E}$. If also $V(\mu, \mathfrak{A}) < +\infty$, then $\int_T\mu$ exists.*

Proof. It can easily be shown [3; 17] that the filter generated by $\{\mathfrak{E}(\mu, \mathbf{S}, E): \mathbf{S} \in \mathfrak{A}\}$ (notation as in Definition 3) is Cauchy and so the theorem follows from Lemma 1.

DEFINITION 6. Let μ be an additive absolutely integrable function on $\mathbf{I}$ with values in a Banach space $\mathbf{F}$. The function $\mu^{\#}: \mathfrak{M}_0(\mu) \to \mathbf{F}$ is defined by $\mu^{\#}(X) = \int_T \chi_X\mu$, $X \in \mathfrak{M}_0(\mu)$.

That $\mu^{\#}$ is defined follows from Theorems 2 and 3.

THEOREM 4. *Let μ be an additive absolutely integrable function on* $\mathbf{I}$ *with values in a Banach space* $\mathbf{F}$. *Then $\mu^{\#}$ is a finitely additive set function on the clan $\mathfrak{M}_0(\mu)$ and $||\mu^{\#}(X)|| \leq \mu^*(X)$, $X \in \mathfrak{M}_0(\mu)$. If in addition $(T, \mathfrak{A}, \mathbf{I})$ is decomposable, $\mu^{\#}$ is a vector-valued measure, i.e., countably additive, on the semitribe $\mathfrak{M}_0(\mu)$.*

Proof. The additivity of $\mu^{\#}$ on $\mathfrak{M}_0(\mu)$ follows from Lemma 1(3); the inequality follows from Lemma 1(2).

If $(T, \mathfrak{A}, \mathbf{I})$ is decomposable and $X = \bigcup_{n=1}^{\infty} X_n$, where $\{X_n\}$ is a disjointed sequence of sets in $\mathfrak{M}_0(\mu)$, then

$$\left|\left|\mu^{\#}(X) - \sum_{n=1}^{N} \mu^{\#}(X_n)\right|\right| = \left|\left|\mu^{\#}\left(X - \bigcup_{n=1}^{N} X_n\right)\right|\right| \leq \mu^*\left(X - \bigcup_{n=1}^{N} X_n\right)$$

and this tends to zero as $N \to \infty$ because μ^* is countably additive on $\mathfrak{M}_0(\mu)$ (Theorem 2). This proves the countable additivity of $\mu^{\#}$. It might be noted as well that the inequality of the theorem shows that $\mu^{\#}$ has "finite variation" [1; 32] on $\mathfrak{M}_0(\mu)$.

THEOREM 5. *Let μ be an additive absolutely integrable function on* $\mathbf{I}$ *with values in a Banach space* $\mathbf{G}$. *If an elementary set E satisfies*

$$V(\mu, A[E](\setminus E)) = V(\mu, A[\setminus E](E)) = 0,$$

then E is μ-measurable, μ^*-measurable (Carathéodory sense), $\mu^*(E) = V(\mu, A(E))$ and $\mu^{\#}(E) = \int_{(E)}\mu$, provided $\mu^*(E) < +\infty$.

Proof. In [4] it is shown under much more general hypotheses that $\mu^*(E) = V(\mu, A(E))$ and that E is μ^*-measurable (Carathéodory sense). It remains to prove that $E \in \mathfrak{M}(\mu)$ and that $\mu^{\#}(E) = \int_{(E)}\mu$.

For the first of these statements choose an $\mathbf{S} \varepsilon \mathfrak{A}$ so that $V(\mu, \mathbf{S}[E](\setminus E)) \leq \epsilon/3$, so that $V(\mu, \mathbf{S}[\setminus E](E)) \leq \epsilon/3$ and so that (2.3) is smaller than $\epsilon/3$ for μ. Consider

$$(\mathbf{D}_1[E] \times \mathbf{D}_2[\setminus E]) \sum \|\mu(I_1 \cap I_2, x_1)\|.$$

These sums are not decreased by insisting that each $\mathbf{D}_1$, $\mathbf{D}_2$ be contained in $\mathbf{S}(E) \cup \mathbf{S}(\setminus E)$ (Use the additivity of μ and Definition 1(1) and (2).).

If we assume this, then

$$\begin{aligned}(\mathbf{D}_1[E] \times \mathbf{D}_2[\setminus E]) &\sum \|\mu(I_1 \cap I_2, x_1)\| \\ &\leq (\mathbf{D}_1[E](E) \times \mathbf{D}_2[\setminus E](E)) \sum \|\mu(I_1 \cap I_2, x_1)\| \\ &\quad + (\mathbf{D}_1[E](\setminus E) \times \mathbf{D}_2[\setminus E](\setminus E)) \sum \|\mu(I_1 \cap I_2, x_1)\| \\ &\leq (\mathbf{D}_1[E](E) \times \mathbf{D}_2[\setminus E](E)) \sum \|\mu(I_1 \cap I_2, x_2)\| \\ &\quad + (\mathbf{D}_1[E](\setminus E) \times \mathbf{D}_2[\setminus E](\setminus E)) \sum \|\mu(I_1 \cap I_2, x_1)\| \\ &\quad + (\mathbf{D}_1[E](E) \times \mathbf{D}_2[\setminus E](E)) \sum \|\mu(I_1 \cap I_2, x_1) - \mu(I_1 \cap I_2, x_2)\| \\ &\leq \epsilon.\end{aligned}$$

Symmetric arguments give the other result (2.2) and so E is μ-measurable as required.

To show that $\mu^{\#}(E) = \int_{(E)} \mu$ we choose $\mathbf{S} \varepsilon \mathfrak{A}$ so that the following hold.

(i) $(\mathbf{D}_1 \times \mathbf{D}_2) \sum \|\chi_E(x_1)\mu(I_1 \cap I_2, x_1) - \chi_E(x_2)\mu(I_1 \cap I_2, x_2)\| \leq \epsilon/5$.

(ii) $V(\mu, \mathbf{S}[E](\setminus E)) \leq \epsilon/5$.

(iii) $V(\mu, \mathbf{S}[\setminus E](E)) \leq \epsilon/5$.

We choose an $E_1 \varepsilon \mathfrak{E}$ so that

(iv) $\|\int_{(E_1)} \chi_E \mu - \int_T \chi_E \mu\| \leq \epsilon/5$

and a partition $\mathbf{D}_1$ of E_1 contained in $\mathbf{S}(E) \cup \mathbf{S}(\setminus E)$ (This is possible by Definition 1 and the additivity of μ.) so that

(v) $\|(\mathbf{D}_1) \sum \chi_E(x_1)\mu(I_1, x_1) - \int_{(E_1)} \chi_E \mu\| \leq \epsilon/5$.

Then if $\mathbf{D}_2$ is any partition of E contained in $\mathbf{S}$ we have

$$\begin{aligned}\left\|(\mathbf{D}_2) \sum \mu(I_2, x_2) - \int_T \chi_E \mu\right\| &\leq \|(\mathbf{D}_2) \sum \mu(I_2, x_2) - (\mathbf{D}_1) \sum \chi_E(x_1)\mu(I_1, x_1)\| \\ &\quad + \left\|(\mathbf{D}_1) \sum \chi_E(x_1)\mu(I_1, x_1) - \int_{(E_1)} \chi_E \mu\right\| \\ &\quad + \left\|\int_{(E_1)} \chi_E \mu - \int_T \chi_E \mu\right\| \\ &\leq 2\epsilon/5 + \|\mathbf{D}_2[E] \sum \mu(I_2, x_2) - \mathbf{D}_1(E) \sum \chi_E(x_1)\mu(I_1, x_1)\| \\ &\quad + \mathbf{D}_2[\setminus E] \sum \|\mu(I_2, x_2)\| + \mathbf{D}_1(\setminus E) \sum \|\chi_E(x_1)\mu(I_1, x_1)\| \\ &\leq 2\epsilon/5 + V(\mu, \mathbf{S}[\setminus E](E)) + V(\mu, \mathbf{S}[E](\setminus E)) \\ &\quad + (\mathbf{D}_2[E] \times \mathbf{D}_1(E)) \sum \|\mu(I_2, x_2) - \chi_E(x_1)\mu(I_1, x_1)\| \\ &\leq \epsilon.\end{aligned}$$

And so it follows from the definition of $\int_{(E)}\mu$ that $\int_{(E)}\mu = \int_T \chi_E \mu = \mu^{\#}(E)$ as required thus completing the proof of the theorem.

Suppose now that **E**, **F** and **G** are normed linear spaces with a bilinear mapping $\mathbf{E} \times \mathbf{F} \rightarrow \mathbf{G}$, denoted $(a, b) = ab$, $a \in \mathbf{E}$, $b \in \mathbf{F}$, satisfying $||ab|| \leq ||a|| \cdot ||b||$. For any functions $f: T \rightarrow \mathbf{E}$ and $\mu: \mathbf{I} \rightarrow \mathbf{F}$ we define the function $f\mu: \mathbf{I} \rightarrow \mathbf{G}$ by $f\mu: (I, x) \rightarrow f(x)\mu(I, x)$.

DEFINITION 7. With the notation as above $\mathfrak{L}_{\mathbf{E}}(\mu)$ denotes the collection of all functions $f: T \rightarrow \mathbf{E}$ such that $f\mu$ is absolutely integrable in $(T, \mathfrak{A}, \mathbf{I})$ and for which $N_\mu(||f||) < +\infty$ (see Theorem 1).

$\mathfrak{L}_{\mathbf{E}}(\mu)$ is clearly a linear space and shall be considered equipped with the topology supplied by the seminorm $f \rightarrow N_\mu(||f||)$.

THEOREM 6. *If* **E** *is a Banach space and* $(T, \mathfrak{A}, \mathbf{I})$ *is decomposable, then* $\mathfrak{L}_{\mathbf{E}}(\mu)$ *is complete and* $N_\mu(||f||) = 0$ *if and only if* $f(x) = 0$, μ^**-almost everywhere.*

Proof. See [5; Theorem 7] and [2; 524].

THEOREM 7. *Let* **G** *be a Banach space and suppose that* μ *is additive. Then* $\int_T f\mu$ *exists for every* $f \in \mathfrak{L}_{\mathbf{E}}(\mu)$ *and* $f \rightarrow \int_T f\mu$ *is a continuous linear mapping from* $\mathfrak{L}_{\mathbf{E}}(\mu)$ *into* **G**.

Proof. See Theorem 3 and Lemma 1.

3. Integration with respect to finitely additive measures. More specific results are obtainable when the function μ on **I** is independent of $x \in T$ and so in effect is defined on the clan $\mathfrak{B}$. In fact we will write $\mu(I) = \mu(I, x)$, $I \in \mathfrak{B}$, in this case and consider an additive function on **I** as a finitely additive measure on $\mathfrak{B}$. Some warning should be taken however. The integral $\int_T f\mu$ defined with respect to μ is not the traditional one defined for finitely additive measures; for decomposable systems $(T, \mathfrak{A}, \mathbf{I})$ it is the countably additive measure $\mu^{\#}$ that is the underlying measure. Theorem 9 gives an instance when these coincide.

DEFINITION 8. Let μ be a function on **I** with values in **F**, where **E**, **F** and **G** are as in the previous section, and independent of $x \in T$. A function $f: T \rightarrow \mathbf{E}$ is said to be *strongly absolutely* μ*-integrable* if for every $\epsilon > 0$ there is an $\mathbf{S} \in \mathfrak{A}$ such that

$$(\mathbf{D}_1 \times \mathbf{D}_2) \sum ||f(x_1) - f(x_2)|| \cdot ||\mu(I_1 \cap I_2)|| \leq \epsilon$$

for every pair of partitions $\mathbf{D}_1$ and $\mathbf{D}_2$ contained in **S**.

Note that if f is strongly absolutely μ-integrable, then $f\mu$ is absolutely integrable and so Definition 8 is a stronger requirement than Definition 5.

Let $\mathfrak{L}^{*}_{\mathbf{E}}(\mu)$ denote the collection of all $f: T \rightarrow \mathbf{E}$ which are strongly absolutely μ-integrable and for which $N_\mu(||f||) < +\infty$. If **E** is the scalar field, then $\mathfrak{L}^{*}_{\mathbf{E}}(\mu)$ and $\mathfrak{L}_{\mathbf{E}}(\mu)$ coincide.

LEMMA 3. *Let* $(T, \mathfrak{A}, \mathbf{I})$ *be decomposable and suppose* $\mathbf{E}$ *is a Banach space. Then* $\mathfrak{L}^*_{\mathbf{E}}(\mu)$ *is a closed subspace of* $\mathfrak{L}_{\mathbf{E}}(\mu)$.

Proof. A slight modification of the proof of [5; Theorem 7] shows $\mathfrak{L}^*_{\mathbf{E}}(\mu)$ is complete and the lemma thus follows.

THEOREM 8. *Let* $(T, \mathfrak{A}, \mathbf{I})$ *be a decomposable partitioning system and suppose that* $\mathbf{E}$, $\mathbf{F}$ *and* $\mathbf{G}$ *are Banach spaces and that* $\mu:\mathbf{I} \to \mathbf{F}$ *is additive and independent of* $x \in T$. *Then every* f *in* $\mathfrak{L}^*_{\mathbf{E}}(\mu)$ *is measurable with respect to the tribe* $\mathfrak{M}(\mu)$. *Conversely if* $\mathbf{E}$ *is separable, every* $\mathfrak{M}(\mu)$*-measurable function* f *with* $N_\mu(||f||) < +\infty$ *belongs to* $\mathfrak{L}^*_{\mathbf{E}}(\mu)$ *and the integral* $\int_T f\mu$ *is equal to the integral* $\int f(x)\, d\mu^*(x)$ *with respect to the vector-valued measure* μ^*.

Proof. The proofs in [3; 35] are valid here.

THEOREM 9. *Let* $\mu:\mathbf{I} \to \mathbf{F}$ *be additive and independent of* $x \in T$ *and suppose* $\mathbf{F}$ *is a Banach space. If for every set* $E \in \mathfrak{E}$, $\mu^*(E) < +\infty$ *and*

$$V(\mu, \mathfrak{A}[E](\setminus E)) = V(\mu, \mathfrak{A}[\setminus E](E)) = 0,$$

then $\mathfrak{E}$ *is contained in* $\mathfrak{M}_0(\mu)$ *and* μ^* *is an extension of* μ.

Proof. See Theorem 5.

We conclude this section by remarking on the following problem. If $f \in \mathfrak{L}^*_{\mathbf{E}}(\mu)$, then $||f|| \in \mathfrak{L}(||\mu||) = \mathfrak{L}_{\mathbf{R}}(||\mu||)$, this corresponding to the "absolute integrability" of every $f \in \mathfrak{L}^*_{\mathbf{E}}(\mu)$. However as $||\mu||$ is in general not additive Theorem 7 does not yield the existence of the integral $\int_T ||f|| \cdot ||\mu||$.

This can be circumvented as follows. The functional $f \to N_\mu(f)$ defined for $f \in \mathfrak{L}(||\mu||)$ with $f \geq 0$ is additive ([5; Theorem 6] can be proved in our context for subadditive set functions.) and so in a unique way can be extended to a function on all of $\mathfrak{L}(||\mu||)$ (as in [5]) and this we shall call the integral $\int_T f \cdot ||\mu||$. This integral is consistent with the integral defined in §1.

This supplies the following classical result in a simpler form. If $f \in \mathfrak{L}^*_{\mathbf{E}}(\mu)$, then $||f|| \in \mathfrak{L}(||\mu||)$ and

$$\left\|\int_T f\mu\right\| \leq \int_T ||f|| \cdot ||\mu||.$$

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