THE NATURAL INTEGRAL ON THE REAL LINE

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Dedicated to Ralph Henstock (1923–2007).

1 Introduction This essay is a highly personal revisiting of the integral definition that Ralph Henstock discovered and the methods that he introduced. Ralph's research program largely focussed on summability theory and nonabsolute integration, the latter becoming his major interest for the last several decades. Neither of these topics is much in vogue today, so it is particularly ironic that he has achieved a measure of fame, even immortality, as a result of something he noticed while investigating some ideas in nonabsolute integration theory.

This thing that he apparently stumbled across (at the same time as the Czech mathematician Jaroslav Kurzweil in Prague) is a simple definition for integration theory on the real line. A definition itself does not usually have such an impact unless the discoverer then goes on to pioneer many of the necessary insights from the definition. But this definition (both simple and trivial) is for the integration theories of Lebesgue and Denjoy. Those integrals had properties already well known. What it meant, merely, is that one could replace the Lebesgue definition of the integral (and Denjoy's definition of countably many integrals) with a single appealing definition and then develop the analysis that we associate with those two great mathematicians in a different and simpler order.

For example it is much easier to have the Lebesgue integral already defined and then show how the measure theory constructs that integral than to use the measure theory to *define* an integral, which integral then must be developed from the involved construction. Indeed, whatever reluctance mathematicians in 1901 had to Lebesgue's methods were mostly focussed on the highly nonintuitive definition and the arduous nature of the techniques needed to establish the simplest properties. If this statement is true about Lebesgue's integral it is surely more so for Denjoy's integral, whose methods would have seemed truly baroque when first proposed to a world not so eager even to follow Lebesgue.

We shall sketch out the formal treatment of the natural integral with a view to understanding its structure. In this we are guided by Henstock's program for his *division spaces.* We avoid any axiomatic treatment of integration theory, but follow the same spirit of the program by describing the natural integral within a framework which is easy to generalize and can be tailored for many different settings.

2 The integral While nearly all readers now will be familiar with the original definition of the integral uncovered by Henstock and Kurzweil back in about 1957, it is worthwhile giving it in full, if only to point out the remarkable simplicity and economy.

The entirety of elementary integration theory on the real line can be captured by the following few lines:

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For a function $f : [a,b] \to \mathbb{R}$ the *integral*,

$$\int_a^b f(x)\,dx,$$

is defined by the requirement that

for every $\epsilon > 0$ and every $x \in [a,b]$ there is a positive number $\delta(x)$ with the property that

$$\left| \int_a^b f(x)\,dx - \sum_{k=1}^{n} f(\xi_k)(a_k - a_{k-1}) \right| < \epsilon$$

for any subdivision

$$a = a_0 < a_1 < a_2 < \dots a_{n-1} < a_n = b$$

with points satisfying

$$a_{k-1} \le \xi_k \le a_k \text{ and } 0 < a_k - a_{k-1} < \delta(\xi_k) \quad (k = 1, 2, \dots, n).$$

- f is *integrable* if this statement holds for some real number $\int_a^b f(x)\,dx$.
- f is *uniformly integrable* [Riemann integrable] if this statement holds moreover with a uniform choice of $\delta(x)$.
- f is *absolutely integrable* [Lebesgue integrable] if both f and $|f|$ are integrable.
- f is *nonabsolutely integrable* if only f but not $|f|$ is integrable.

No other definitions for the basic integral on the real line are needed. Special integrals based on the approximate derivative or the symmetric derivative, or integrals using convergence factors such as the Cesàro-Perron series of integrals would need further techniques and methods, but for the vast majority of users of integration on the real line no other definitions should be introduced. Certainly the traditional Riemann and Lebesgue integrals are already included here and need no definition of their own.

The student would, naturally enough, be reminded here of the analogy in language with infinite series. A series $\sum_{k=1}^{\infty} a_k$ is said to be absolutely convergent if both $\sum_{k=1}^{\infty} |a_k|$ and $\sum_{k=1}^{\infty} a_k$ converge. The series is nonabsolutely convergent if $\sum_{k=1}^{\infty} |a_k|$ diverges while $\sum_{k=1}^{\infty} a_k$ converges. The absolutely convergent series have the most robust theory, the broadest range of applications, and receive the most attention. A similar assertion holds for the absolutely integrable functions, i.e. those that are Lebesgue integrable.

There are now numerous accounts of the integral suitable for exposition to the uninitiated. Henstock's own first account in [14] was not successful, although his paper [16] is quite lucid. Recommended for this purpose are [1], [2], [3], [9], [24], [27], [29], [30], [32], [38], and [45].

2.1 Goodbye to the Riemann integral This definition effectively kills off the Riemann integral. To be sure, Riemann integrable functions remain as an interesting subclass of the integrable functions, but there is no Riemann integral left to discuss. There is just "the integral."

Replace all calculus definitions of the Riemann integral with this definition in a typical presentation and nothing is lost except that the theory becomes the honest, modern theory of integration on the real line. Many simplifications are possible because this definition is, astonishingly perhaps, easier to work with than the usual definition of a Riemann integral as a uniform limit of Riemann sums. This case is made in [38].

Here are two instances, but there are many more. The fundamental theorem of the calculus,

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

is proved in freshman calculus classes by invoking the mean-value theorem and adding, of course, the hypothesis that F' is Riemann integrable. In the theory here the integrability of F' is proved (not assumed *a priori*) and the only tool needed is the definition of the derivative. The proof is shorter, stronger, and more elementary. The second instance is the proof that a continuous function is integrable. This can be obtained easily using just the definition of what continuity means. On the contrary, for the calculus course, one must prove the stronger statement that a continuous function is uniformly integrable (i.e., Riemann integrable). This requires first proving that continuous functions on a compact interval are uniformly continuous, an onerous requirement at the early level.

2.2 "The Lebesgue integral is dead." I hesitate to write this line since it is so easily subject to misinterpretation, not to mention hostility. The quotation is from Henstock himself who used it in an address at an international meeting many years ago. I don't know the content of his talk but I can give my own exegesis.

Once the correct definition of the integral on the real line is accepted, "Lebesgue's integral" disappears in the sense that his *definition* disappears. In its place is something rather more beautiful and powerful. We could call it now *Lebesgue's theorem*: the absolutely integrable functions are characterized as those measurable functions for which a certain simple measure-theoretic construction of the integral is possible.

The historical fact that Lebesgue used that measure theory to define his integral is actually unfortunate. There was much resistance to his integral merely based on the arcane nature of its definition. There would surely have been universal and instant acceptance of his integral had Lebesgue given this Riemann sum definition first and then characterized his integral in the manner we are all familiar with.

There will be no loss if Lebesgue's definition for the integral disappears and is replaced by his deep analysis of the measure properties of integrals instead. At the graduate level, naturally, one shows how this analysis can be used to develop an integration theory on any measure space.

2.3 Denjoy and Perron integrals? The definitions of Denjoy and Perron also disappear into the mathematical dustbin. This integral now has too compelling and simple a definition for one to consider starting at either of the notions of Denjoy and Perron, even though their definitions characterize the same integral.

The great contribution of Denjoy can be redescribed this way. He gives a transfinite hierarchy of classes of integrable functions that show exactly how constructive the integral is for the large class of nonabsolutely integrable functions. But we have no need for a countable number of integrals, each extending the previous in a transfinite sequence.

2.4 Nomenclature Every advertising and marketing student learns that the choice of a name for a product is an essential part of the process of getting an item accepted broadly. As explained in the Wikipedia site, for example, a good brand name should (among other things) be easy to pronounce, be easy to remember, be easy to recognize, suggest product benefits, and distinguish the product's positioning relative to the competition.

The integral as we have defined it has been marketed under some truly unfortunate names, names which have inhibited its acceptance by the mathematical community. Henstock originally called it the *Riemann complete integral* and, later, the *generalized Riemann integral*. More recently the integral has been called by some *the gauge integral* or *gage integral*, wherein the intention is to give the $\delta(x)$ that appears in the definition some prominence as a gauge and use that to identify it. Many authors call it the *Henstock integral*, the *Kurweil integral*, or the *Henstock-Kurweil integral*. It is described sometimes as a *Riemann-type* integral. We could even go back to the term *Denjoy-Perron integral* or *Denjoy totalization* for names with an early 20th century appeal. All of these names can be held partially responsible for the reluctance of the academic mathematical community to introduce this integration theory into the undergraduate curriculum.

The currently dominating brands are the Riemann integral and the Lebesgue integral. Even the latter has some avoidance built in because of the perception that the theory is too difficult for undergraduate mathematics students, and certainly much too difficult for engineering students. In the mathematical market today nearly everyone would consider any "new" integral different from these as arcane, unusual, baroque, too specialized, esoteric, or recondite. Certainly such qualities, while they might attract some experts, should require that we keep such an integral far away from our students.

To change this perception I would propose that this integration theory might be called the *natural integral on the real line*.

3 The structure of the natural integral on the line The ϵ-δ definition of the integral given in Section 2 does not much reveal the structure of the theory. Its only advantage seems to be that, by formulating the integral as if it were a minor adjustment to the Riemann integral definition, this might lead to an easier acceptance of the theory. As we now know, half-a-century later, this is not the case: it remains for most a curiosity.

Henstock reworked the theory into an abstract structure in a series of papers [12], [13], and [15]. None of these are particularly readable and they seem to be without influence. Following a suggestion from one of his students that he might try for a more compelling axiomatic treatment (similar to measure spaces) he produced [18] which contains an uncharacteristically readable and compelling account of his ideas and methods using the structure he called a division space.

There has been little study of any of Henstock's abstract structure but there is nonetheless interesting information in the structure theory itself. One way to see this is to reformulate the definition of the natural integral in such a way that the path to a generalization is relatively obvious. The theory that follows is a more elaborate account of the natural integral in which certain aspects of the theory are presented in ways that could be used for reformulations in different settings.

This is not so much an abstraction of the methods (as one finds in studies of metric spaces or measure spaces) but merely an invitation to use *similar* structures as needed for other purposes.

3.1 Covering relations A collection β is a *covering relation* if it consists of pairs $([u,v],w)$ where $[u,v]$ is a compact interval and w is a point contained in that interval. All covering relations are thus merely subsets of the fundamental set[1]

$$\mathbb{H} = \{([u,v],w) : u,v,w \in \mathbb{R},\ u \le w \le v,\ u < v\}.$$

The following language is essential.

[1]Henstock's original version of the integral used an endpoint version so that only pairs $([u,v],w)$ where $w=u$ or $w=v$ are considered. This does have advantages, as Leader [21] and [22] also points out.

- For any set E of real numbers and any $\beta \subset \mathbb{H}$
$$\beta[E] = \{([u,v],w) \in \beta : w \in E\}.$$
- For any set E of real numbers and any $\beta \subset \mathbb{H}$
$$\beta(E) = \{([u,v],w) \in \beta : [u,v] \subset E\}.$$
- A finite covering relation
$$\pi = \{([u_i,v_i],w_i) : i = 1,2,\ldots,n\}$$
is a *partition* of an interval $[a,b]$ if the intervals
$$\{[u_i,v_i] : i = 1,2,\ldots,n\}$$
do not overlap and their union is all of the interval $[a,b]$.
- Any subset of a partition is a *subpartition.*
- β is a *full cover* of a set E if for every $u \in E$ there is a $\delta > 0$ so that β contains all pairs of the form
$$([s,t],u) \ \text{ for all } t - s < \delta \text{ and } u \in [s,t]\ .$$
- β is a *fine cover* of a set E if for every $u \in E$ and every $\delta > 0$ there is at least one pair $([s,t],u) \in \beta$ for which $t - s < \delta$ and $u \in [s,t]$.

A full cover (fine cover) without mention of a set E would mean a full cover (fine cover) of all of $\mathbb{R}$.

The family of all full covers is denoted $\mathcal{H}$. The family of all fine covers is considered as a dual to $\mathcal{H}$ and so is denoted in the suggestive way $\mathcal{H}^*$. The following fundamental property of $\mathcal{H}$ is what makes integration theory possible:

> $\mathcal{H}$ is a filter on $\mathbb{H}$ *with the property that every element* $\beta \in \mathcal{H}$ *contains a partition of every compact interval.*

3.2 The natural integral The upper and lower integrals of a function $f : [a,b] \to \mathbb{R}$ are defined as upper and lower limits of Riemann sums where the limits are taken with regard to the filter $\mathcal{H}$.

$$\overline{\int_a^b} f(x)\,dx = \inf_{\beta\in\mathcal{H}} \sup_{\pi\subset\beta} \sum_{([u,v],w)\in\pi} f(w)(v-u)$$
$$\underline{\int_a^b} f(x)\,dx = \sup_{\beta\in\mathcal{H}} \inf_{\pi\subset\beta} \sum_{([u,v],w)\in\pi} f(w)(v-u)$$

where π denotes an arbitrary partition of $[a,b]$ chosen from the given covering relation β.

If the upper and lower integrals agree then we say that the integral is *determined* and write the common value as
$$\int_a^b f(x)\,dx.$$
If this common value is finite we say, moreover, that f is *integrable.* If both f and $|f|$ are integrable we say that f is *absolutely integrable.* If f is integrable but $|f|$ is not then we say that f is *nonabsolutely integrable.*

3.3 Other variants As Henstock was quick to point out, none of this theory requires the integrand to have such a traditional form and the Riemann sums

$$\sum_{([u,v],w)\in\pi} f(w)(v-u)$$

can easily be replaced by sums of the form

$$\sum_{([u,v],w)\in\pi} h([u,v],w)$$

for any suitable function h defined on $\mathbb{H}$. The space of values assumed by h requires, too, only that some kind of sum is defined and either an order (as here) or a topology be present. Thus the theory is quite flexible and can be tailored to many different purposes. There is now a large literature for integrals of this type with values in Banach spaces (or even topological semigroups).

The same applies to considering integrals of the Stieltjes or Hellinger variety. Thus integrals of the form

$$\int_a^b f(x)\,dG(x) \quad \text{or} \quad \int_a^b \frac{dG(x)dF(x)}{dH(x)}$$

are easily defined by the same methods. Because the methods are the same interconnections are immediate. For example, the identity

$$\int_a^b f(x)\,dG(x) = \int_a^b f(x)G'(x)\,dx$$

is usually hard to work with when one side is interpreted as a Lebesgue integral and the other as a Riemann-Stieltjes integral. If both sides are defined using the same process the connection is nearly trivial.

3.4 Structure of derivatives The filter $\mathcal{H}$ and its companion dual $\mathcal{H}^*$ are intimately related to the process of differentiation on the real line. Thus one structure describes both integration and differentiation of real functions. It is for that reason that the connections between the two concepts of differentiation and integration are so intimate. From this structural point of view this intimacy is obvious. In Lebesgue's theory of integration, it is not at all clear at the beginning how long it will make the connection between differentiation and integration.

Let $F : \mathbb{R} \to \mathbb{R}$. Then the upper and lower derivates of F at a given point x can be defined in terms of the filter $\mathcal{H}$ as

$$\overline{D}F(x) = \inf_{\beta\in\mathcal{H}} \sup_{([u,v],x)\in\beta} \frac{F(v)-F(u)}{v-u}$$

$$\underline{D}F(x) = \sup_{\beta\in\mathcal{H}} \inf_{([u,v],x)\in\beta} \frac{F(v)-F(u)}{v-u}$$

$$\overline{D}F(x) = \sup_{\beta\in\mathcal{H}^*} \inf_{([u,v],x)\in\beta} \frac{F(v)-F(u)}{v-u}$$

$$\underline{D}F(x) = \inf_{\beta\in\mathcal{H}^*} \sup_{([u,v],x)\in\beta} \frac{F(v)-F(u)}{v-u}$$

Derivatives have another expression in terms of these families $\mathcal{H}$ and $\mathcal{H}^*$:

$F'(x) = f(x)$ at every point in a set E if and only if for every $\epsilon > 0$ the family

$$\beta = \{([u,v],x) : |F(v) - F(u) - f(x)(v-u)| \leq \epsilon(v-u)\}$$

is a full cover of E.

And, again dually,

$F'(x) = f(x)$ at every point in a set E if and only if for some positive function $\epsilon : E \to \mathbb{R}$ the family

$$\beta = \{([u,v],x) : |F(v) - F(u) - f(x)(v-u)| > \epsilon(x)(v-u)\}$$

is not fine at any point of E.

3.5 Structure of the variational measures Let $F : \mathbb{R} \to \mathbb{R}$ and let β be any covering relation. The *variation* of F relative to β is defined as

$$V(F,\beta) = \sup_{\pi\subset\beta} \sum_{([u,v],w)\in\pi} |F(v) - F(u)|$$

where π denotes any subpartition contained in the covering relation β. This variation leads to two *variational measures* associated with the function F on a set $E \subset \mathbb{R}$:

$$V^*(F,E) = \inf_{\beta\in\mathcal{H}} V(F,\beta[E])$$

and

$$V_*(F,E) = \inf_{\beta\in\mathcal{H}^*} V(F,\beta[E]).$$

We refer to these as the *full and fine variations* of F on the set E.

The special case $F(x) = x$ leads to two versions of the Lebesgue measure

$$\mathcal{L}^*(E) = \inf_{\beta\in\mathcal{H}} \sup_{\pi\subset\beta[E]} \sum_{([u,v],w)\in\pi} (v-u)$$

and

$$\mathcal{L}_*(E) = \inf_{\beta\in\mathcal{H}^*} \sup_{\pi\subset\beta[E]} \sum_{([u,v],w)\in\pi} (v-u)$$

which we call the *full* and *fine versions* of Lebesgue measure (in fact the full and fine variations of the identity function $F(x) = x$ on the set E.)

A further variational measure is also useful to add to the account. Let $f : \mathbb{R} \to \mathbb{R}$ and define

$$V(f(x)\,dx,\beta) = \sup_{\pi\subset\beta} \sum_{([u,v],w)\in\pi} |f(w)|(v-u)$$

where π denotes any subpartition contained in the covering relation β. If $E \subset \mathbb{R}$ then we write

$$V^*(f(x)\,dx,E) = \inf_{\beta\in\mathcal{H}} V(f(x)\,dx,\beta[E])$$

and

$$V_*(f(x)\,dx,E) = \inf_{\beta\in\mathcal{H}^*} V(f(x)\,dx,\beta[E])$$

and refer to these as the *full and fine outer integrals* of f on the set E. These measures play the role of the expression

$$\int_E |f(x)|\,dx$$

that would be used in Lebesgue's theory, but now not restricted to measurable f or to measurable E.

3.6 The nature of the duality The fine covers (which the reader no doubt recognizes as Vitali covers) play a role that is dual to the full covers. The duality can be expressed by the following statements. Here we say that a covering relation β ignores a point x if β contains no pairs of the form $([u,v],x)$.

> *A subset $\beta \subset \mathbb{H}$ is a fine cover of a set E if and only if the intersection $\beta \cap \beta_1$ ignores no point of E for every β_1 that is a full cover of E.*

And the dual version:

> *A subset $\beta \subset \mathbb{H}$ is a full cover of a set E if and only if the intersection $\beta \cap \beta_1$ ignores no point of E for every β_1 that is a fine cover of E.*

The essential feature of many Vitali covering arguments on the real line can be captured by the following language and accompanying theorems (see [44]).

Definition 1 *A function $F : \mathbb{R} \to \mathbb{R}$ is said to have the* Vitali property *on a set E if the full and fine variations of F agree on all subsets of E, i.e.,*

$$V^*(F,A) = V_*(F,A) \text{ for all } A \subset E.$$

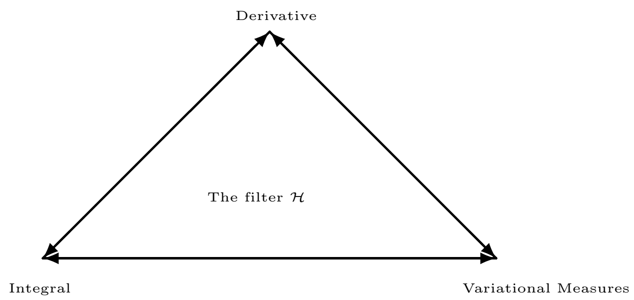
3.1 (Vitali covering theorem) *If the function $F : \mathbb{R} \to \mathbb{R}$ is continuous and locally of bounded variation then F has the Vitali property on every subset of $\mathbb{R}$.*

Corollary 3.2 (Vitali covering theorem) *The full and fine versions of Lebesgue's measure on the real line are identical, i.e., $\mathcal{L}^*(E) = \mathcal{L}_*(E)$ for every set $E \subset \mathbb{R}$.*

Notice that the latter statement is a special case of the former and is the one usually given this name. See [44] for details.

3.7 The fundamental program The program for developing the integration theory can be considered to be that required by finding the interrelations among the these three fundamental concepts (derivative, integral, variational measure) each of which is defined directly from the filter $\mathcal{H}$. The fact that there are such relations is transparent: since each of the three concepts is defined by the same structure the relationships are nearly immediate.

This triangle visualizes the structure of the theory:



But this should be considered the purely formal aspects of the theory. Neither the integration nor the measure theory is constructive in any sense, because the full covers that are needed in theory are not constructive themselves. Thus the program is completed by finding more constructive methods to discuss these concepts. Lebesgue's measure theory (which is constructive) is needed in order to provide a different access to the measures $\mathcal{L}^*$ and $\mathcal{L}_*$. That measure theory then provides, too, a constructive account of the integral for absolutely integrable functions. Denjoy's constructive totalization process completes Lebesgue's construction by showing how to handle the nonabsolutely integrable functions, whose theory using only $\mathcal{H}$ is again strictly formal.

3.8 The fundamental theorem of the calculus As an illustration of the general program from Section 3.7 let us recount the form in which the fundamental theorem of the calculus can be given for this integral.

> *(A) Let $f : [a, b] \to \mathbb{R}$ and suppose that f is integrable on $[a, b]$ with $F(t) = \int_a^t f(x)\,dx$ $(a \le t \le b)$. Then $F'(t) = f(t)$ for $\mathcal{L}_*$-almost every t in $[a, b]$.*

> *(B) Let $F, f : [a, b] \to \mathbb{R}$ and suppose that $F'(t) = f(t)$ for every t in $[a, b]$ excepting possibly a set N for which $\mathcal{L}^*(N) = V^*(F, N) = 0$. Then f is integrable on $[a, b]$ and $F(t) = \int_a^t f(x)\,dx$ $(a \le t \le b)$.*

These two theorems are completely elementary and merely require formal arguments directly from the definitions. This is possible because all the concepts in the two statements have definitions directly in terms of the full and fine covers. Arguing among the concepts of derivative, integral, and variational measures presents no unusual barriers.

In the standard presentation of Lebesgue's integral the fundamental theorem is, in stark contrast, considerably deeper and very much nonelementary. Moreover, in the Lebesgue theory, the fundamental theorem of the calculus is also incomplete and weaker than that given here. There (A) would have to start with the stronger assumption that f is absolutely integrable and requires advance knowledge of the Vitali covering theorem so that the fine measure $\mathcal{L}_*$ can be replaced by Lebesgue's measure. Equally bad, assertion (B) requires an added assumption that f is absolutely integrable or that F is absolutely continuous.

This material offers a compelling account of why, starting with the definition of the natural integral, one can more rapidly develop the outlines of the major points of integration theory. At the end, add in Lebesgue's measure-theoretic characterization of the absolutely integrable functions and you have all the usual graduate material on integration on the real line, done in a simpler and more transparent manner.

Most graduate courses prefer to treat integration on the line as a special application of the general theory of measure and integration. This is a legitimate point of view, certainly, but it misses a chance to introduce this attractive subject at an earlier level and skips over some very interesting analysis of real functions.

4 Henstock division spaces While Henstock's future in mathematics history is surely secured by his joint discovery of the simple definition of the natural integral on the real line, his real research interest was mostly in the broader theory of integration. The structure he proposed in [18] and named as *division spaces* has not been taken up as a topic of interest. His goal of providing a framework which would describe nearly all processes of integration, including those using convergence factors, made for an interesting research project but one in which he was mostly a solitary traveler. The apparent fate for his abstract theories has been that the methods are now used freely throughout the research literature on this topic, without, however, his main goal of producing a structure that (like measure spaces) would inform the direction of the field.

4.1 Integration bases Let us follow, briefly, his suggestion for an abstract presentation. A first step in generalizing the situation for the natural integral would be to completely imitate the structure that we found there. If we remain on the real line then the following definition suggests itself.

Definition 2 *A family $\mathcal{I}$ of subsets of $\mathbb{H}$ is said to be an* integration basis *on $\mathbb{R}$ if*

1. *$\mathcal{I}$ is a filter on $\mathbb{H}$, and*

2. *for every compact interval* $[a, b]$ *and every element* $\beta \in \mathcal{I}$ *there is a subset* $\pi \subset \beta$ *that is a partition of* $[a, b]$.

Given two integration bases $\mathcal{I}_1$ and $\mathcal{I}_2$ we say that $\mathcal{I}_1$ is *finer* than $\mathcal{I}_2$ if the former filter is finer than the latter, namely if $\mathcal{I}_1 \supset \mathcal{I}_2$. Thus the collection of integration bases on the line is partially ordered in the obvious way by set inclusion. Note that it is also inductively ordered.

It should be obvious how one could use an integration basis to define integrals, derivatives, and variational measures. In some cases a dual basis can be introduced to some benefit. We do not give the details. Indeed it seems without merit to try to develop a general theory, if only because no one seems interested in pursuing such a study. The best way to promote such a study seems, instead, simply to encourage one to use the methods and formal language of the natural integration theory to inform the theory that one wishes to develop.

For example, an account of the approximate Perron integral, based on the approximate derivative, can be given that closely parallels the theory for the natural integral. Henstock's program was to give a general abstract theory that would *include* the approximate Perron integral as a special case. Since the abstract theory requires rather too much devotion and too many details to construct, it is far simpler to encourage students of the approximate Perron integral to formulate the integral in the correct language and to compare and contrast the development with that for the natural integral.

4.2 Examples The collection of all full covers of the real line we have denoted $\mathcal{H}$. This is certainly an integration basis according to our definition (in fact it is our model) and it describes the natural integral. The following examples show that, even with such little ambition as we are showing here, there is an abundance of interesting things to say.

Riemann integral We say a covering relation β is a *uniformly full cover* of the real line if there is a $\delta > 0$ so that β contains all pairs of the form

$$([s, t], u) \ \text{ for all } t - s < \delta \text{ and } u \in [s, t] \ .$$

The collection of all uniformly full covers of the real line is denoted $\mathcal{R}$ and describes the usual Riemann integral.

Refinement integral Let $C \subset \mathbb{R}$ be a finite set. We define a covering relation β_C that *refines* C by writing

$$\beta_C = \{([s, t], u) : \ u \in [s, t] \text{ and } (s, t) \cap C = \emptyset\}.$$

The collection of all covers β_C for C finite is denoted $\mathcal{R}_0$ can be used to describe the usual Darboux integral.

Original Henstock integral Define the following subset of $\mathbb{H}$:

$$\mathbb{H}_0 = \{([u, v], w : \ w = u \text{ or } w = v\}.$$

The filter on $\mathbb{H}$ generated by the filter base $\{\beta \cap \mathbb{H}_0 : \beta \in \mathcal{H}\}$ is denoted $\mathcal{H}_0$ and also describes the usual Henstock-Kurzweil integral, but in the endpoint-version that Henstock preferred.

Approximate Perron integral The collection of all approximate full covers of the real line is denoted $\mathcal{A}$ and describes the approximate Perron integral. (See [19], [41]).

Maximal integral Let $\mathcal{U}$ denote any integration basis on the reals that contains $\mathcal{H}_0$ and is maximal. Since the family of integration bases is inductively ordered, such an integration basis must exist (by Zorn's lemma).

McShane's integral Define the following set that properly contains $\mathbb{H}$:

$$\mathbb{H}_{\#} = \{([u, v], w) : \ u, \ v, \ w \in \mathbb{R}, \ u < v\}.$$

We say a set $\beta \subset \mathbb{H}_{\#}$ is a *McShane full cover* of the real line if for every x there is a $\delta > 0$ so that β contains all pairs of the form

$$([s, t], u) \ \text{ for all } [s, t] \subset (u - \delta, u + \delta) \ .$$

The collection of all McShane full covers of the real line, denoted $\mathcal{M}$, is a filter on $\mathbb{H}_{\#}$. This violates very slightly our definition of an integration basis, but an integration, differentiation and variational measure theory can be developed in very much the same way. The resulting integral is exactly the Lebesgue integral (see [27]). McShane's book [28] reveals his real motivation for introducing this variant of the natural integral.

There are others which follow exactly this pattern and can be considered integration bases in the narrow sense here. But, to capture some other important situations, modifications are needed. We have already indicated that McShane's scheme requires a slightly enlarged version of what a covering relation is. For integrals which invert symmetric derivatives or approximate symmetric derivatives families of covering relations can be used as here, but not necessarily satisfying exactly these two axioms for an integration basis. See [33], [37], [41] for symmetric full covers and [34] for approximate symmetric full covers.

An interesting series of papers has appeared using dyadic covers to study integrals and derivatives along the lines sketched here. For example see [7], [**?**], and [46].

The problem of recovering a continuous function from one of its Dini derivatives was discussed in Lebesgue's monograph [23]. A scheme using covering relations, modeled in a way similar to the methods outlined in the present paper, appears in [11]. It is based on a covering argument (that is a clever twist on Vitali covers) introduced by Hagood [10] that he applied to give a simple proof of the Lebesgue differentiation theorem.

In higher dimensions there is much more geometry to exploit and harder questions to pose. There are now many kinds of theory possible and numerous interesting directions; a MathSci search for contributions of Washek Pfeffer will track down nearly all of these. Certainly start with his contribution [32] in this issue. See, in particular, his two monographs [30] and [31].

In measure spaces a recent paper of Boccuto and Skvortsov [4] study structures of this type. A search should reveal many more. Similarly in topological spaces Boccuto and Riečan [5] impose a similar structure and develop a theory of integration and differentiation there.

The literature is surprisingly large, although the level of contribution is uneven. Thus, even though Henstock's original plan of a single structure that could describe all these different special integrals, his general intention will have, it seems, a lasting impact on this subject. My own hope is that a unified language (using covering relations, variational measures, the fine/full duality, etc.) will develop and be accepted, making the expositions in this field have a degree of unity. While the division space structure that Henstock hoped would serve this purpose will likely not survive, his methods can be found in nearly all of the details.

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