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Henstock-Kurzweil integrals on time scales

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Abstract

A definition for a version of the Henstock-Kurzweil integral on time scales is given using covering arguments. The integral is shown to be expressible, in some situations, as an ordinary integral in the Newton, the Lebesgue, and Henstock-Kurzweil senses.

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1 Introduction

The Henstock-Kurzweil integral has recently (see [2] and [13]) been introduced into the setting of time scale research in the expectation that it will prove useful there. That integral is a purely formal notion that does not have the same transparency or constructibility as the Riemann and Lebesgue integrals. But the formalism has many advantages. The simplest and most compelling one that readers initially encounter is the manner in which this formalism allows the immediate integration of exact derivatives. This is merely because the formalism itself is nearly identical to the definition of the derivative. For the Riemann and Lebesgue integrals, since their definitions are quite remote from the formal definition of a derivative, it is quite a chore to relate derivatives and integrals; more severely, those integrals fall short in being able to integrate all derivatives anyway.

Whether this formalism of the Henstock-Kurzweil integral will be of service to time scale researchers depends on the depth of the applications that are yet to be found. In this article we relate the time scale version of the integral to the usual form. This relation shows that most of the properties of a time scale integral can be realized by using the techniques of ordinary integration on the real line and do not require special techniques tailored to the time scale setting.

One can still give a definition in the language of the time scale concepts, but proofs need not be confined to arguments within that structure where conventional real analysis arguments would supply the same information.

2 Time scales

A *time scale* is simply a nonempty, closed subset $\mathbb{T}$ of the real numbers. We will keep this fixed throughout the discussion. Note that $\mathbb{T}$ is also completely described by the sequence of contiguous intervals

$$\{[a_i, b_i] : i = 1, 2, 3, \dots\}$$

together with the bounds, if any, of $\mathbb{T}$. Recall that $[a, b]$ would be a *contiguous interval* to $\mathbb{T}$ if both endpoints a and b belong to $\mathbb{T}$ but

$$\mathbb{T} \cap (a, b) = \emptyset.$$

(The sequence may be finite or even empty, but we will write it always as if there are infinitely many intervals $[a_i, b_i]$.)

I cannot detect whether time scale researchers use any standard and consistent language to refer to the contiguous intervals, but we can here call them the *gaps*. A time scale is characterized either by announcing the set $\mathbb{T}$ or announcing the sequence of gaps and the bounds of $\mathbb{T}$ if it has bounds. The time scale literature is more likely to refer directly to the gaps using the language

$$\{[t, \sigma(t)] : t \in \mathbb{T},\ \sigma(t) > t\}$$

which is rather more mysterious to the uninitiated. (See below for the definition of the forward jump operator σ.)

Since an upper bound of $\mathbb{T}$ (if it has one) plays a different role in some discussions many researchers use the expression

$$\mathbb{T}^\kappa = \{t \in \mathbb{T} : t < \sup \mathbb{T}\}$$

to refer to the time scale with that point removed.

The language of time scales uses the following special terminology:

- The *forward jump* operator is the function $\sigma : \mathbb{T} \to \mathbb{T}$ defined by $\sigma(a_i) = b_i$ for all i and $\sigma(t) = t$ for all $t \in \mathbb{T}$ that are not a right-hand endpoint of a contiguous interval. (Note that if $t_0 = \sup \mathbb{T}$ is finite then this definition requires that $\sigma(t_0) = t_0$ which is the usual convention.)
- A function $F : \mathbb{T} \to \mathbb{R}$ is *continuous* (continuous at a point) if it is continuous in the usual relative sense (i.e., using the topology that $\mathbb{T}$ inherits as a subset of $\mathbb{R}$).

- The set of points $\{a_i\}$ from $\mathbb{T}$ is called the *right-scattered* points. The set of points $\{b_i\}$ from $\mathbb{T}$ is called the *left-scattered* points.
- A function $F : \mathbb{T} \to \mathbb{R}$ is said to have a *delta derivative* $F^\Delta(t)$ at a point $t \in \mathbb{T}$ provided that for every $\epsilon > 0$ there is a neighborhood U of t so that

$$\left|[F(\sigma(t) - F(s)] - F^\Delta(t)\,(\sigma(t) - s)\right| \leq \epsilon\,|\sigma(t) - s|$$

for all $s \in U$.

A moment's reflection on the definition of the delta derivative shows that this derivative exists if and only if F is continuous at t and that, if $t = a_i$ for some i then

$$F^\Delta(a_i) = \frac{F(b_i) - F(a_i)}{b_i - a_i},$$

while if t is not a right-scattered point (i.e., is not one of the a_i) then the derivative $F'(t)$ exists (taken relative to the set $\mathbb{T}$) and

$$F^\Delta(t) = F'(t).$$

(If t is an upper bound for $\mathbb{T}$ and isolated in $\mathbb{T}$ then any number would serve as a delta derivative, but would play no role in any discussion here.)

We add to this language by proposing two natural extensions of functions that are defined on time scales. These definitions are not in the spirit of the time scale research program since they make explicit reference to the structure of the particular time scale. But they are useful in translating problems in time scales back to problems in ordinary integration and differentiation, allowing the usual arguments to be used. The first extension can also be found in [8] where it plays a similar role.

Definition 1. *If $f : \mathbb{T} \to \mathbb{R}$ is a function on the time scale $\mathbb{T}$ then a function $f_\flat : \mathbb{R} \to \mathbb{R}$ is defined by*

$$f_\flat(t) = f(t) \quad (t \in \mathbb{T}),$$

$$f_\flat(s) = f(a_i) \quad (a_i < s < b_i),$$

and $f(s) = 0$ elsewhere. (We call this the flat extension *of f since it extends the function to be level on all the gaps.)*

Definition 2. *If $F : \mathbb{T} \to \mathbb{R}$ is a function on the time scale $\mathbb{T}$ then a function $F_\natural : \mathbb{R} \to \mathbb{R}$ defined by*

$$F_\natural(t) = f(t) \quad (t \in \mathbb{T}),$$

and

$$F_\natural(s) = F(a_i) + \frac{[s - a_i][F(b_i) - F(a_i)]}{b_i - a_i} \quad (a_i < s < b_i).$$

If one or both of $t_0 = \sup \mathbb{T}$ or $t_{-1} = \inf \mathbb{T}$ is finite then we would set $F_\natural(s) = F(t_0)$ $(s > t_0)$ and $F_\natural(s) = F(t_{-1})$ $(s < t_{-1})$. (We call this function the natural extension *of F since it extends the function to be linear on all the gaps.)*

3 Covering relations

The formal relation between the ordinary derivative and the Henstock-Kurzweil integral can be expressed in the language of covering relations. We recall some of that language first, then we consider what the same natural application of these ideas in the setting of time scales must be.

A collection β is a *covering relation* if it consists of pairs $([u, v], w)$ where $[u, v]$ is a compact interval and w is an endpoint[1] of that interval. For any set E of real numbers, and any covering relation β one often needs to "prune" away members of β:

$$\beta[E] = \{([u, v], w) \in \beta : w \in E\}.$$

A finite covering relation

$$\pi = \{([u_i, v_i], w_i) : i = 1, 2, \ldots, n\}$$

is a *partition* of an interval $[a, b]$ if the intervals

$$\{[u_i, v_i] : i = 1, 2, \ldots, n\}$$

do not overlap and their union is all of the interval $[a, b]$. Any subset of a partition would be called a *subpartition.* The phrase π is a subpartition of $[a, b]$ would mean that each $([u_i, v_i], w_i)$ belonging to π would have $[u_i, v_i]$ a subinterval of $[a, b]$.

Definition 3. *We say a covering relation β is a* full cover *if for every t in $\mathbb{R}$ there is a $\delta > 0$ so that β contains all pairs of the form*

$$([s, t], t) \quad \textit{for all } t - \delta < s < t$$

and

$$([t, s], t) \quad \textit{for all } t < s < t + \delta.$$

The key lemma for both the integration theory and the differentiation theory on the real line is Cousin's covering lemma, easily proved (see, for example Bartle [3] or [4].).

Lemma 3.1 (Cousin)**.** *Every full cover contains a partition of any compact interval $[a, b]$.*

The family of all full covers forms a filter [a differentiation basis] which can be used to describe both integrals and derivatives. The connection between full covers and derivatives is immediate. The assertion that $F'(x) = f(x)$ for all x in a set E is equivalent to the requirement that for every $\epsilon > 0$ there is a full cover β for which

$$\beta[E] \subset \{([u, v], w) : |F(v) - F(u) - F'(w)(v - u)| < \epsilon(v - u)\}.$$

[1] The usual presentation allows w to be *any* point on the interval $[u, v]$. Henstock's original version of the integral used the endpoint version and that is the version discussed here. The covers of [14], for example, are of the former type.

The connection between full covers and integrals is immediate too: a function $f : \mathbb{R} \to \mathbb{R}$ is *integrable [Henstock-Kurzweil sense]* on an interval $[a, b]$ if and only if for every $\epsilon > 0$ there is a full cover β so that

$$\left| \sum_{([u,v],w)\in\pi} f(w)(v-u) - \int_a^b f(x)\,dx \right| < \epsilon$$

for every partition π of $[a, b]$ contained in β. The much celebrated ease of connecting derivatives and integrals in the Henstock-Kurzweil integral setting arises from nothing deeper than the covering relations connecting the two ideas.

The majority of the published presentations of this theory express themselves in the unfortunate language of "gauges." The full covers offer a structure that explains the theory quite transparently and provide a technical language that is convenient to use and manipulate. The gauge language is clumsier and generalizes must less easily. Peterson and Thompson [13] have adopted the gauge language to the time scale setting; we prefer to use the covering language, believing it offers a clearer expression of the underlying ideas.

4 Descriptive characterizations of the Henstock-Kurzweil integral

Since our intention is to discuss the version of the Henstock-Kurzweil integral appropriate to time scales, it will be useful to recall some of the theory of the usual version. The parallels are very close and very natural. The methods are identical.

Lemma 4. *A set E of real numbers is a null set [i.e., a set of Lebesgue measure zero] if and only if for every $\epsilon > 0$ we may select a full cover β with the property that*

$$\sum_{([u,v],w)\in\pi} (v-u) < \epsilon$$

for all subpartitions π that are contained in $\beta[E]$.

Definition 5. *A function $F : \mathbb{R} \to \mathbb{R}$ is said* not to grow *on a set E of real numbers if, for every $\epsilon > 0$, we may select a full cover β with the property that*

$$\sum_{([u,v],w)\in\pi} |F(v) - F(u)| < \epsilon$$

for all subpartitions π that are contained in $\beta[E]$.

The two main characterizations of the integral now follow. The first is due to Henstock and is a key technical tool in handling this kind of integral. The second describes the integral as a Newton type integral, as an antiderivative. It lies a little deeper but can easily be proved using the Henstock criterion and a Vitali covering argument. Details of all of these ideas can be found in Smithee [14], expressed in a similar language.

Theorem 6 (Henstock Criterion)**.** *A function $f : \mathbb{R} \to \mathbb{R}$ is integrable [Henstock-Kurzweil sense] on an interval $[a, b]$ if and only if there is a continuous function $F : \mathbb{R} \to \mathbb{R}$ with the property that for every $\epsilon > 0$ there is a full cover β so that*

$$\sum_{([u,v],w)\in\pi} |f(w)(v-u) - [F(v) - F(u)]| < \epsilon$$

for every partition π of $[a, b]$ contained in β. In that case

$$\int_a^b f(t)\, dt = F(b) - F(a).$$

Theorem 7 (Descriptive Characterization)**.** *A function $f : \mathbb{R} \to \mathbb{R}$ is integrable [Henstock-Kurzweil sense] on an interval $[a, b]$ if and only if there is a continuous function $F : \mathbb{R} \to \mathbb{R}$ and a null set E with the property that $F'(x) = f(x)$ for all x in $[a, b]$ excepting possibly at points in E and F does not grow on E. In that case*

$$\int_a^b f(t)\, dt = F(b) - F(a).$$

5 Covering arguments on time scales

A similar language is available in the time scale setting merely by designing the covers to yield the delta derivative in the same way the ordinary full covers yield the ordinary derivative. We have designed the language to exactly mimic the usual language, thus allowing nearly identical arguments to be used.

Definition 8. *We say a covering relation β is a full cover in $\mathbb{T}$ if for every t in $\mathbb{T}$ there is a $\delta > 0$ so that β contains all pairs of the form*

$$([s, t], t) \quad \text{for all } t - \delta < s < t \text{ and } s \in \mathbb{T})$$

and

$$([t, s], t) \quad \text{for all } t < s < t + \delta \text{ and } s \in \mathbb{T})\ .$$

Definition 9. *The collection*

$$\beta_g = \{([a_i, b_i], a_i) : i = 1, 2, 3, \dots\}$$

is called the forward gap cover*.*

Definition 10. *We say a covering relation β is a $\mathbb{T}$-full cover if it is a full cover in $\mathbb{T}$ that includes the forward gap cover.*

The definition of a $\mathbb{T}$-full cover is designed to accommodate, not characterize, the delta derivative. Suppose that a function $F : \mathbb{T} \to \mathbb{R}$ has a delta derivative $F^{\Delta}(t) = f(t)$ at every point. Let $\epsilon > 0$. Then, certainly, for every $w \in \mathbb{T}$ that is not a right-scattered point, there is a $\delta(w) > 0$ so that

$$|F(v) - F(u) - f(w)(v - u)| < \epsilon(v - u)$$

whenever u and v are points of $\mathbb{T}$ for which $0 < v-u < \delta$. At the right-scattered points a_i select $0 < \delta(a_i) < b_i - a_i$ so that

$$\delta(a_i) < \frac{\epsilon 2^{-i}}{|f(a_i)|+1},$$

and so that

$$|F(u) - F(a_i)| < \epsilon 2^{-i}$$

whenever $0 < a_i - u < \delta_i$. This just exploits the continuity of F at the point a_i.

Define β to be the collection of pairs $([u,v],w)$ for which u, v, $w \in \mathbb{T}$, $0 < v-u < \delta(w)$. This is a full cover in $\mathbb{T}$. Throw in the forward gap cover β_g and remember that

$$|F(v) - F(u) - f(w)(v-u)| = 0$$

for all $([u,v],w) \in \beta_g$.

Then $\beta \cup \beta_g$ becomes a $\mathbb{T}$-full cover. Observe now that if $\pi \subset \beta \cup \beta_g$ is a partition of an interval with endpoints in $\mathbb{T}$,

$$\sum_{([u,v],w)\in\pi} |F(v) - F(u) - f(w)(v-u)| \leq \epsilon \sum_{([u,v],w)\in\pi} (v-u) + 2\epsilon \leq \epsilon(b-a+2).$$

Consequently we have exactly the setting for a Henstock-Kurzweil type integral that inverts derivatives since

$$\left| F(b) - F(a) - \sum_{([u,v],w)\in\pi} f(w)(v-u) \right|$$

$$\leq \sum_{([u,v],w)\in\pi} |F(v) - F(u) - f(w)(v-u)| \leq \epsilon(b-a+2).$$

To make this work only requires now the assurance that such partitions exists, a version of Cousin's lemma. The two versions of Cousin's lemma have nearly identical proofs. Peterson and Thompson [13] write out the details showing that the usual nested-interval proof of Cousin's lemma, with minor alterations, can be used for the time scale version.

Lemma 5.1 (Cousin's Lemma)**.** *Every $\mathbb{T}$-full cover contains a partition of any compact interval $[a,b]$ whose endpoints are in $\mathbb{T}$.*

6 Newton integral

The earliest use of an integration theory on time scales simply used the Newton integral. Let us recall the ordinary Newton integral on the real line. The version here, more useful in the calculus than more familiar and less general versions, allows a countable exceptional set (as suggested in [14] and [15]).

Definition 11. *A function $f : \mathbb{R} \to \mathbb{R}$ is said to be* integrable (Newton sense) *on an interval $[a, b]$ provided that there is a continuous function $F : \mathbb{R} \to \mathbb{R}$ such that*

$$F'(t) = f(t)$$

at every point t in the open interval (a, b) with perhaps countably many exceptions. In that case we write

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

The corresponding time scale version is much the same. The justification for both integrals is the fact that if two functions F and G can both be verified to satisfy the conditions of the definition then $F(b) - F(a) = G(b) - G(a)$. A covering argument for the ordinary version appears in [14] and an exact copy of that argument (but using Lemma 5.1 rather than Lemma 3.1) will handle the time scale version.

Definition 12. *A function $f : \mathbb{T} \to \mathbb{R}$ is said to be* delta integrable (Newton sense) *on an interval $[a, b]$ with endpoints in $\mathbb{T}$ provided that there is a continuous function $F : \mathbb{T} \to \mathbb{R}$ such that*

$$F^{\Delta}(t) = f(t)$$

at every point t in the interval $[a, b)$ excepting perhaps countably many non right-scattered points. In that case we write

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

We show that the Newton version of the delta integral can be reduced to an ordinary Newton integral.

Theorem 13. *Suppose that $f : \mathbb{T} \to \mathbb{R}$ and that $[a, b]$ is an interval with endpoints in $\mathbb{T}$. Then f is delta integrable (Newton sense) on $[a, b]$ if and only if $f_\flat$ is integrable (Newton sense) on $[a, b]$. Also,*

$$\int_a^b f(t)\,\Delta t = \int_a^b f_\flat(s)\,dx$$

7 Integral of Peterson and Thompson

For time scale integrals we have the following version of a Henstock-Kurzweil integral which is a natural extension to the time scale setting.

Definition 14. *A function $f : \mathbb{T} \to \mathbb{R}$ is* delta integrable [Peterson-Thompson sense] *on a compact interval $[a, b]$ with endpoints in $\mathbb{T}$ and has value $\int_a^b f(t)\,\Delta t$ provided that for every $\epsilon > 0$ there is a $\mathbb{T}$-full cover β so that*

$$\left| \sum_{([u,v],w)\in\pi} f(w)(v-u) - \int_a^b f(t)\Delta t \right| < \epsilon$$

for all partitions π of the interval $[a,b]$ that are contained in β.

This is the integral defined in Peterson and Thompson[2][13]. The definition there is somewhat different and expressed in the language of gauges but easily checked to yield the same integration theory as this definition. We believe that, while the theory is unchanged, the present language should make this work more accessible; in particular, rather than hiding the technical details in the peculiarities of the gauge, the covering language clarifies what is happening.

8 Descriptive version of the Peterson-Thompson integral

Descriptive characterizations are also possible for the Peterson-Thompson integral. The first connection, identical to Theorem 6, is always available in any Henstock type theory provided that there are partitions as supplied here by Lemma 5.1. This is also cited in [13, Theorem 2.15] where it is correctly pointed out that the usual proofs in the ordinary integration transfer with only minor changes to the time scale setting.

Theorem 15 (Henstock Criterion). *A function $f : \mathbb{T} \to \mathbb{R}$ is delta integrable [Peterson-Thompson sense] on a compact interval $[a,b]$ with endpoints in $\mathbb{T}$ if and only if there is a continuous function $F : \mathbb{T} \to \mathbb{R}$ so that for every $\epsilon > 0$ there is a $\mathbb{T}$-full cover β so that*

$$\sum_{([u,v],w)\in\pi} |F(v) - F(u) - f(w)(v-u)| < \epsilon$$

for all partitions π of the interval $[a,b]$ that are contained in β. In that case necessarily

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

The other descriptive version uses derivatives, null sets and the concept of a function that does not grow on null sets. Again the standard methods can be repeated here with minor alterations. (For example [14] has a proof.) Here we simply report the necessary definitions and state the theorem that is the direct analog of Theorem 7.

Definition 16. *A function $F : \mathbb{T} \to \mathbb{R}$* does not $\mathbb{T}$-grow *on a set $E \subset \mathbb{T}$ provided that for every $\epsilon > 0$ there is a $\mathbb{T}$-full cover β so that*

$$\sum_{([u,v],w)\in\pi} |F(v) - F(u)| < \epsilon$$

for all subpartitions π of the interval $[a,b]$ that are contained in $\beta[E]$.

[2]The B. Thompson of the cited paper is not the B. Thomson who is the author of the present paper although the confusion about these spellings is universal.

Definition 17. *A set $E \subset \mathbb{T}$ is a $\mathbb{T}$-null set if for every $\epsilon > 0$ there is a $\mathbb{T}$-full cover β so that*

$$\sum_{([u,v],w)\in\pi} (v-u) < \epsilon$$

for all subpartitions π of the interval $[a,b]$ that are contained in $\beta[E]$.

[Remark: Equivalently $E \subset \mathbb{T}$ is a $\mathbb{T}$-null set if and only if E contains no right-scattered points and E has Lebesgue measure zero as a subset of $\mathbb{R}$. Note that E is a $\mathbb{T}$-null set if and only if the function $F(t) = t$ does not $\mathbb{T}$-grow on E.]

Theorem 18 (Descriptive Characterization)**.** *A function $f : \mathbb{T} \to \mathbb{R}$ is delta integrable [Peterson-Thompson sense] on a compact interval $[a,b]$ with endpoints in $\mathbb{T}$ if and only if there is a continuous function $F : \mathbb{T} \to \mathbb{R}$ and a $\mathbb{T}$-null set E, such that F does not $\mathbb{T}$-grow on E, and such that $F^{\Delta}(t) = f(t)$ for all points in $[a,b] \cap \mathbb{T}$ excepting possibly at points of the set E. In that case necessarily*

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

Note that it is a corollary of Theorem 18 that the delta integral [Peterson-Thompson sense] includes the Newton integral as given in Definition 13. This is merely because a continuous function $F : \mathbb{T} \to \mathbb{R}$ cannot $\mathbb{T}$-grow on any countable set of non right-scattered points. This is also proved in [13, Theorem 2.3].

9 Representation of the Peterson-Thompson integral

Our main theorems in this section show, in many cases, that the delta integral just defined can be directly reduced to more familiar objects. Displaying the delta integral as a conventional integral on the real line allows one to use ordinary arguments to obtain integration properties rather than detailed considerations within the time scale setting. That does not detract from the general program of the time scales analysis, which is to derive a unified theory; it merely allows some proofs to be translated over to the real line setting and unnecessary complications avoided.

The theorems all assert a relation between the delta integral of a function $f : \mathbb{T} \to \mathbb{R}$ and the usual integral of its extension $f_\flat : \mathbb{R} \to \mathbb{R}$. Proofs appear in the later sections.

Theorem 19. *Suppose that $f : \mathbb{T} \to \mathbb{R}$ and that $[a,b]$ is an interval with endpoints in $\mathbb{T}$. Then f is delta integrable (Peterson-Thompson sense) on $[a,b]$ if the corresponding function $f_\flat$ is integrable in the Henstock-Kurzweil sense on $[a,b]$. Then, necessarily,*

$$\int_a^b f(t)\,\Delta t = \int_a^b f_\flat(s)\,dx.$$

There is a partial converse available if we add in an assumption about the behavior of the integral on the gaps.

Theorem 20. *Suppose that $f : \mathbb{T} \to \mathbb{R}$, that $[a, b]$ is an interval with endpoints in $\mathbb{T}$, and suppose that f is delta integrable (Peterson-Thompson sense) on $[a, b]$. Suppose that*

$$\sum_{a_i \in [a,b)} |f(a_i)|(b_i - a_i) < \infty. \tag{1}$$

Then the corresponding function $f_\flat$ is integrable in the Henstock-Kurzweil sense on $[a, b]$ and

$$\int_a^b f(t)\,\Delta t = \int_a^b f_\flat(s)\,dx.$$

Corollary 21. *Suppose that $f : \mathbb{T} \to \mathbb{R}$, that $[a, b]$ is an interval with endpoints in $\mathbb{T}$, and suppose that f is absolutely delta integrable (Peterson-Thompson sense) on $[a, b]$, i.e., that the integrals*

$$\int_a^b f(t)\,\Delta t \quad \text{and} \quad \int_a^b |f(t)|\,\Delta t$$

both exist in this sense. Then the corresponding function $f_\flat$ is integrable in the Lebesgue sense on $[a, b]$ and

$$\int_a^b f(t)\,\Delta t = \int_a^b f_\flat(s)\,dx = \int_{\mathbb{T}\cap[a,b]} f(t)\,dt + \sum_{a_i \in [a,b)} f(a_i)(b_i - a_i)$$

where the series is absolutely convergent.

The two theorems and the corollary do not present a complete picture of the relationships here. We do not know whether the integrability of $f : \mathbb{T} \to \mathbb{R}$ necessarily implies the integrability of the extension $f_\flat : \mathbb{R} \to \mathbb{R}$. The additional assumption (1) in the theorem is needed for our method of proof but perhaps can be dropped. If we are less ambitious in Theorem 20 then we could use a more general integral. It is easy enough to tailor a Henstock-Kurzweil integral to handle such a situation; the Denjoy-Khintchine integral also works in this context. We do not offer such material since a reasonable conjecture is that integrability of f in Theorem 20 is alone enough to require that $f_\flat$ is integrable in the Henstock-Kurzweil sense on $[a, b]$.

10 Proofs

10.1 Proof of Theorem 13.

Proof. To prove one direction let us assume that f is delta integrable on $[a, b]$. Then there must be a continuous function $F : \mathbb{T} \to \mathbb{R}$ and a countable set C containing no right-scattered points such that

$$F^\Delta(t) = f(t)$$

at every point t in $[a,b)\setminus C$ and

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

Consider the natural extension of F to $F_\natural$ and the flat extension of f to $f_\flat$. Note that $F_\natural$ is continuous since F is.

Observe that if $[a_i, b_i] \subset [a,b]$ then

$$f(a_i) = F^\Delta(a_i) = \frac{F(b_i) - F(a_i)}{b_i - a_i}$$

which happens to be also the slope of the linear segment of the graph of $F_\natural$ on the interval $[a_i, b_i]$.

Thus we see that the derivative $F_\natural'(s) = f_\flat(s)$ at all points $a_i < s < b_i$. At any point t in $(a,b)\cap\mathbb{T}$ but not in C nor a right-scattered point the relative derivative $F'(t) = f(t)$. It is easy to check that this means that $F_\natural'(t) = f(t)$ there too.

The continuous function $F_\natural$ now has the property that it has a finite derivative $F_\natural'(s) = f_\flat(s)$ at every point $s \in (a,b)$ with the exception of points in C or right-scattered points. By definition, then, $f_\flat$ has a Newton integral and, in that sense,

$$\int_a^b f_\flat(s)\,ds = F_\natural(b) - F_\natural(a) = F(b) - F(a) = \int_a^b f(t)\,\Delta t.$$

This completes the proof in this direction.

Conversely suppose that $f_\flat$ is Newton integrable in the sense defined. Then there is a continuous function $G:\mathbb{R}\to\mathbb{R}$ and a countable set C so that $G'(s) = f_\flat(s)$ for every $a < s < b$ that does not belong to C. Let F denote the restriction of G to $\mathbb{T}$. This function $F:\mathbb{T}\to\mathbb{R}$ is continuous.

Note that, because $f_\flat$ is constant on each interval $[a_i, b_i)$ contained in $[a,b]$ the function G must be linear on any such interval and, indeed,

$$f(a_i) = \frac{G(b_i) - G(a_i)}{b_i - a_i} = \frac{F(b_i) - F(a_i)}{b_i - a_i}.$$

In particular for the points a_i that belong also to the interval $[a,b)$ it must be the case that

$$F^\Delta(a_i) = f(a_i).$$

Let C_0 denote the set of points in C that are not right-scattered. At every point $t \in \mathbb{T}\cap[a,b)$ not belonging to C_0 we note that

$$f(t) = f_\flat(t) = G'(t) = F^\Delta(t).$$

Consequently F is a primitive for f in the appropriate Newton sense and f must be delta integrable (Newton sense). Moreover,

$$\int_a^b f(t)\Delta t = F(b) - F(a) = G(b) - G(a) = \int_a^b f_\flat(s)\,ds.$$

■

10.2 Proof of Theorem 19.

Proof. Suppose that $f_\flat$ is Henstock-Kurzweil integrable on $[a, b]$. Let $\epsilon > 0$. Then we may select a full cover β with the property that

$$\left| \sum_{([u,v],w)\in\pi} f_\flat(w)(v-u) - \int_a^b f_\flat(t)\,dt \right| < \epsilon/2$$

for all partitions π of the interval $[a, b]$ that are contained in β.

Let us construct a new covering relation β' from β. We include in β' all elements from β, but removing those elements $([u, v], w)$ for which u, v and w do not all belong to $\mathbb{T}$. It is immediate that β' is a full cover in $\mathbb{T}$. If we add in the forward gap cover we arrive at a covering relation $\beta' \cup \beta_g$ that is necessarily a $\mathbb{T}$-full cover.

Let π' be any partition of $[a, b]$ that is contained in $\beta' \cup \beta_g$ and consider the Riemann sum

$$\sum_{([u,v],w)\in\pi'} f(w)(v-u).$$

Each element $([u, v], w) \in \pi'$ either belongs to the original covering relation β or else

$$([u, v], w) = ([a_i, b_i], a_i)$$

for some i. In the latter case we will replace that pair by a partition π_i of $[a_i, b_i]$ chosen from the original cover β. Notice that, since $f_\flat$ is constant on $[a_i, b_i)$ the partition π_i can be chosen so that

$$\left| \sum_{([u,v],w)\in\pi_i} f_\flat(w)(v-u) - f(a_i)(b_i - a_i) \right| < \epsilon 2^{-i-1}.$$

Thus we let π'' denote the partition consisting of all elements in the original partition π' but with the removal of elements that belong to the forward gap cover, i.e., for which

$$([u, v], w) = ([a_i, b_i], a_i)$$

for some i, and the replacement of each such element by the elements of the partition π_i.

We find that

$$\sum_{([u,v],w)\in\pi'} f(w)(v-u)$$

and

$$\sum_{([u,v],w)\in\pi''} f_\flat(w)(v-u)$$

differ by less than $\epsilon/2$. From that we deduce that

$$\left| \sum_{([u,v],w)\in\pi'} f(w)(v-u) - \int_a^b f_\flat(t)\,dt \right| < \epsilon$$

for all partitions π' of the interval $[a, b]$ that are contained in the $\mathbb{T}$-full cover $\beta' \cup \beta_g$.

By definition, then, f is delta integrable (Peterson and Thompson sense) on $[a, b]$ and its delta integral is given by the value of the Henstock-Kurzweil integral

$$\int_a^b f_\flat(t)\, dt$$

as we wished to prove. ■

10.3 Proof of Theorem 20

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Proof. Suppose that f is delta integrable (Peterson and Thompson sense) on $[a, b]$. Then we can use the Henstock criterion of Theorem 15 and the descriptive characterization of Theorem 18 to claim the existence of a continuous function $F : \mathbb{T} \to \mathbb{R}$ with the properties stated in those theorems.

Note that

$$f(a_i)(b_i - a_i) = F(b_i) - F(a_i)$$

for each interval $[a_i, b_i]$ contained inside $[a, b]$. Hence our assumption that

$$\sum_{a_i \in [a,b)} |f(a_i)|(b_i - a) < \infty$$

is equivalent to the statement that

$$\sum_{a_i \in [a,b)} |F(b_i) - F(a_i)| < \infty.$$

From Theorem 18 we know that there is a $\mathbb{T}$-null set $E \subset \mathbb{T}$ so that $F'(t) = f(t)$ for every point t in $[a, b] \cap \mathbb{T}$ excepting possibly at the points of E. And we know that F does not $\mathbb{T}$-grow on E. Define E_1 to be the set E with all right-scattered points added in. The set E_1 is a null set (set of Lebesgue measure zero).

We introduce the natural extension of F, the function $F_\natural : \mathbb{R} \to \mathbb{R}$, and note that it is continuous (since F is continuous). We observe that $F_\natural'(t) = f_\flat(t)$ for every point t in $[a, b] \cap \mathbb{T}$ excluding possibly the points in the null set E_1. But it is trivial that $F_\natural'(s) = f_\flat(s)$ for all s in any interval (a_i, b_i) contained in $[a, b]$. Thus we have really that $F_\natural'(s) = f_\flat(s)$ for every s in $[a, b]$ except possibly at the points of the set E_1.

We claim that $F_\natural$ does not grow on E_1. Consequently Theorem 6 will allow us to conclude the integrability (Henstock-Kurzweil sense) of $f_\flat$ on $[a, b]$ and the identity of the theorem would then follow.

Let $\epsilon > 0$ and choose a $\mathbb{T}$-full cover β so that

$$\sum_{([u,v],w) \in \pi} |F(v) - F(u)| < \epsilon \tag{2}$$

for all subpartitions π of the interval $[a,b]$ chosen from $\beta[E]$. Because β is a $\mathbb{T}$-full cover, we may define $\delta_1(t) > 0$ for each $t \in \mathbb{T}$, chosen so

1. β contains all pairs $([s,b_i],a_i)$ provided only that $s \in \mathbb{T}$ and $0 \le a_i - s < \delta_1(a_i)$.
2. If u is in $\mathbb{T}$ but not a right-scattered point then $v \in \mathbb{T}$ and $0 < v-u < \delta_1(u)$ imply that $([u,v],u) \in \beta$.
3. If v is in $\mathbb{T}$ but not a right-scattered point then $u \in \mathbb{T}$ and $0 < v-u < \delta_1(v)$ imply that $([u,v],v) \in \beta$.

Choose m large enough so that

$$\sum_{i=m}^{\infty} |F(b_i) - F(a_i)| < \epsilon.$$

For each t in $\mathbb{T}$ that is not a right-scattered or left-scattered point choose $\delta_2(t) > 0$ so that the open interval

$$(t - \delta_2(t), t + \delta_2(t))$$

contains no points of any interval $[a_i,b_i]$ for which $i < m$.

For each $i = 1,2,3,\dots$ define positive numbers $\delta(a_i)$ and $\delta(b_i)$ chosen so that

$$|F_\natural(v) - F_\natural(u)| < \epsilon 2^{-i-2}$$

provided only that $u \le a_i \le v$ and $v-u < \delta(a_i)$ or $u \le b_i \le v$ and $v-u < \delta(b_i)$. This just takes advantage of the continuity of the function $F_\natural$. For all remaining points $t \in \mathbb{T}$ define

$$\delta(t) = \min\{\delta_1(t), \delta_2(t)\}.$$

Define the covering relation β_1 to include all $([u,v],u)$ for which $u \in \mathbb{T}$, $v \in \mathbb{R}$, and $0 < v-u < \delta(u)$ and also all $([u,v],v)$ for which $v \in \mathbb{T}$, $u \in \mathbb{R}$ and $0 < v-u < \delta(v)$. We insist, too, that β_1 includes any other elements $([u,v],w)$ for which $[u,v]$ is a subinterval of one of the contiguous intervals $[a_i,b_i]$ and $w = u$ or $w = v$ is a point from the interior (a_i,b_i).

This collection β_1 is a full cover. To check this simply verify the condition of the definition at each single point.

Consider any subpartition π_1 of the interval $[a,b]$ chosen from $\beta_1[E_1]$. We shall obtain the estimate

$$\sum_{([u,v],w)\in\pi_1} |F_\natural(v) - F_\natural(u)| < 6\epsilon. \tag{3}$$

Let us break the sum into several parts: if $([u,v],w) \in \pi_1$ then possibly $w = a_i$ or b_i for some index i (if not, note that $w \in E$). Case 1: In the case where $w = a_i$ or b_i for some index i, we would necessarily have

$$|F_\natural(v) - F_\natural(u)| < \epsilon 2^{-i-2}.$$

There could be at most two such pairs for any given i. The total contribution to the sum (3) from all Case 1 possibilities is smaller than

$$\sum_{i=1}^{\infty} 4\epsilon 2^{-i-2} = \epsilon.$$

Now we consider the remaining contributions to the sum (3). These will come from pairs $([u,v],w) \in \pi_1$ for which $w \in E$. For example suppose, Case 2, that $([u,v],u) \in \pi_1$ so that $u \in E$ and that v is also a point of $\mathbb{T}$. Then we have simply

$$|F_\natural(v) - F_\natural(u)| = |F(v) - F(u)|$$

where $([u,v],w) \in \beta$. [Note here this is in the orginal $\mathbb{T}$-full cover β.] A similar consideration applies, Case 3, if $([u,v],v) \in \pi_1$ with $v \in E$ and for which $u \in \mathbb{T}$. Again

$$|F_\natural(v) - F_\natural(u)| = |F(v) - F(u)|$$

where $([u,v],w) \in \beta$. The total contribution of all Case 2 and Case 3 pairs to the sum cannot exceed ϵ because of the inequality (2). This is because the collection of Case 2 and 3 pairs forms a subpartition contained in $\beta[E]$.

There remains only two further possibilities. There may yet remain pairs $([u,v],u) \in \pi_1$ for which $u \in E$ but v is not in $\mathbb{T}$, and also pairs $([u,v],v) \in \pi_1$ for which $v \in E$ but u is not in $\mathbb{T}$.

For such pairs we handle this way. Case 4: if $([u,v],u) \in \pi_1$, $u \in E$ but v is not in $\mathbb{T}$, then v must appear in one of the intervals (a_i, b_i) but only (because of the construction of β_1) for values of $i \geq m$. For that pair we observe that

$$|F_\natural(v) - F_\natural(u)| \leq |F(a_i) - F(u)| + |F(b_i) - F(a_i)|.$$

The pair $([u,a_i],u)$ belongs to β and so we have an estimate for sums of such pairs. The total sum of the possibilities $|F(b_i) - F(a_i)|$ has already been determined to be smaller than ϵ. We conclude that the Case 4 possibilities contribute no more than 2ϵ to the sum.

There remains only Case 5 which is similar: if $([u,v],v) \in \pi_1$, $v \in E$ but u is not in $\mathbb{T}$, then u must appear in one of the intervals (a_i, b_i) but only (because of the construction of β_1) for values of $i \geq m$. For that pair we see that

$$|F_\natural(v) - F_\natural(u)| \leq |F(v) - F(b_i)| + |F(b_i) - F(a_i)|.$$

The pair $([b_i,v],v)$ belongs to β and so we have an estimate for sums of such pairs. The total sum of the possibilities $|F(b_i) - F(a_i)|$ has already been determined to be smaller than ϵ. We conclude that the Case 5 possibilities contribute no more than 2ϵ to the sum.

Finally, then, we have established the inequality (3). By definition the function $F_\natural$ does not grow on the null set E_1 and has as its derivative $f_\flat$ at all points in $[a,b]$ outside of E_1. The descriptive characterization of the Henstock-Kurzweil integral supplies the integrability of $f_\flat$ on $[a,b]$ as well as the identity

$$\int_a^b f_\flat(s)\,ds = F_\natural(b) - F_\natural(a) = F(b) - F(a) = \int_a^b f(t)\,\Delta t.$$

■

10.4 Proof of Corollary 21

Proof. The assumption that $|f|$ is delta integrable on $[a,b]$ forces the sum

$$\sum_{a_i \in [a,b)} |f(a_i)|(b_i - a)$$

to converge. Thus we can conclude from Theorem 20 that both $f_\flat$ and $|f_\flat|$ are integrable in the Henstock-Kurzweil sense on $[a,b]$. Usual properties of that integral then apply to conclude that $f_\flat$ must be integrable in Lebesgue's sense. The final identity of the corollary is a routine application of measure-theoretic properties of the Lebesgue integral. ■

11 Some remarks

It is distressing to see the time scale researchers recapitulating the history of integration theory on the real line. Thus there is now a confused mess of integrals, Newton, Riemann, Darboux, Lebesgue, and Henstock-Kurzweil both on the real line and in the time scale setting. The proper solution on the real line is the development of the Henstock-Kurzweil integral in its simplest presentation. This integral is actually simpler than the others and it immediately relates to the Newton program of integration as a kind of antidifferentiation. In the time scale setting the Peterson-Thompson integral is a likely candidate for a simple unified theory.

The representation theorems of Section 9 show that a time scale integral of a function $f : \mathbb{T} \to \mathbb{R}$ can be realized in most situations as a sum of ordinary integrals;

$$\int_a^b f(t)\,\Delta t = \int_a^b f(t)\chi_{\mathbb{T}}(t)\,dt + \int_a^b \left(\sum_{i=1}^{\infty} f(a_i)\chi_{(a_i,b_i)}(t) \right) dt.$$

This was pointed out before in [8]. This means that for most purposes the ordinary theory of integration can be used to prove properties of time scale integrals. It means too that most expositions of the theory are likely better off to present this fact rather than develop an arcane expression of an integral in the time scale language itself.

References

[1] B. Aulbach and L. Neidhart, Integration on measure chains. Proceedings of the Sixth International Conference on Difference Equations, 239–252, CRC, Boca Raton, FL, 2004.

[2] S. Avsec, B. Bannish, B. Johnson, and S. Meckler, The Henstock-Kurzweil delta integral on unbounded time scales. Panamer. Math. J. 16 (2006), no. 3, 77–98.

[3] R. G. Bartle, Return to the Riemann integral, *Amer. Math. Monthly* **103** (1996) 625-632.

[4] ——————, *A Modern Theory of Integration*, Graduate Studies in Math., no. 32, American Mathematical Society, Providence, 2001.

[5] M. Bohner, G. Guseinov, and A. Peterson, *Introduction to the time scales calculus*. Advances in dynamic equations on time scales, 1–15, Birkhäuser Boston, Boston, MA, 2003.

[6] M. Bohner and G. Guseinov, Riemann and Lebesgue integration, in *Advances in dynamic equations on time scales*, 117–163, Birkhäuser, Boston, Boston, MA, 2003.

[7] A. M. Bruckner, J. B. Bruckner and B. S. Thomson, *Real Analysis*, Prentice-Hall (1996).

[8] A. Cabada and D. Vivero, Expression of the Lebesgue Δ-integral on time scales as a usual Lebesgue integral: application to the calculus of Δ-antiderivatives. Math. Comput. Modelling 43 (2006) 194–207.

[9] G. Sh. Guseinov and B. Kaymakçalan, Basics of Riemann delta and nabla integration on time scales, J. Difference Equ. Appl. **8** (2002), no. 11, 1001–1017.

[10] G. Sh. Guseinov, Integration on time scales. J. Math. Anal. Appl. 285 (2003), no. 1, 107–127.

[11] G. Sh. Guseinov and B. Kaymakçalan, On the Riemann integration on time scales, in *Proceedings of the Sixth International Conference on Difference Equations*, 289–298, CRC, Boca Raton, FL, (2004).

[12] R. Henstock, A Riemann-type integral of Lebesgue power, Canad. J. Math. **20** (1968) 79–87.

[13] A. Peterson and B. Thompson, Henstock-Kurzweil delta and nabla integrals, Journal of Mathematical Analysis and Applications, 323 (2006) 162–178.

[14] Alan Smithee, *The Integral Calculus*, Published and available for download on the website **http://www.classicalrealanalysis.com** (2007).

[15] Elias Zakon, *Mathematical Analysis I*, Published and available for download on the website **http://www.trillia.com** (2004).