# DIFFERENTIATION OF INTEGRALS

By

A. M. BRUCKNER

University of California, Santa Barbara

*The Twelfth*

HERBERT ELLSWORTH SLAUGHT

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## CONTENTS

# I. INTRODUCTION[1]

The purpose of this article is to present, in an expository manner, a development of the work done in the last thirty-five years on various questions concerning the differentiation of integrals in spaces more general than the Euclidean line $R_1$. Perhaps the unifying notion would be that of the Fundamental Theorem of Calculus in such spaces. A standard theorem in measure theory is the RADON-NIKODYM THEOREM which states that with appropriate assumptions on a measure space $(X, M, \mu)$, each measure $\sigma$ that is absolutely continuous with respect to $\mu$ can be represented as the integral of some function: $\sigma(E) = \int_E f d\mu$ for all $E \in M$. The function $f$ is often called the "Radon-Nikodym" derivative. But in what sense and under what circumstances is it a derivative in the customary pointwise sense? We consider this question in Chapters 2 and 6. In addition, we consider (i) applications and interpretations of the theory, (ii) a number of items concerning the tools of the theory (covering theorems, density theorems and halo conditions), and (iii) various topics of miscellaneous sorts.

Throughout the first five chapters of this article, we shall be concerned almost entirely with Euclidean $n$-dimensional space $R_n$. We shall use the symbol $\mu$ to denote Lebesgue measure of the appropriate number of dimensions, unless we explicitly state otherwise.

Although some of the material in the sequel can be found in one form or another in certain books, most of it cannot. We list the books Hahn-Rosenthal [49], Haupt-Aumann-Pauc [51], Hayes-Pauc [63], Hewitt-Stromberg [65], McShane [87], Monroe [95], Rudin [137], Saks [144], Shilov-Gurevic [148], Zaanen [167], and Zygmund [170] as ones which deal with the differentiation of integrals in spaces more general than $R_1$. Some of the major works relating to our main Chapters 2 and 6 are Busemann-Feller [21], Denjoy [31], Hayes-Pauc [64], Jessen-Marcinkiewicz-Zygmund [71], Morse [97], Pauc [115], de Possel [126], and Trjitzinsky [154], [158].

While most of our dealings will be with the differentiation of integrals, certain of the ideas carry over to the differentiation of set functions. We occasionally consider such generalizations, but our central theme is the differentiation of integrals, and, accordingly, we have taken that as our title.

For motivation we present Chapter 2 in a less condensed form than the following chapters.

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## II. DIFFERENTIATION OF INTEGRALS IN EUCLIDEAN SPACES

2.1 **Motivation from** $R_1$. For motivation, we begin with a brief discussion of the well-known one-dimensional setting, and cast this setting in a form suitable for generalization. Let $f$ be summable on each compact interval, and define a function $F$ by $F(x) = \int_0^x f\,d\mu$, where $\mu$ denotes one dimensional Lebesgue measure. Then $F'(x) = f(x)$ for almost all $x \in R_1$. This means that $\lim_{h\to 0}[F(x+h)-F(x)]/h = f(x)$ a.e., or equivalently

$$\lim_{h\to 0}\frac{\int_x^{x+h} f\,d\mu}{h} = f(x) \quad \text{a.e.}$$

If we write $\sigma(E) = \int_E f\,d\mu$ for each measurable set $E$ of finite measure, and if we let $I$ denote an interval, then we can use the notation $D\sigma(x) = \lim_{I\Rightarrow x}\sigma(I)/\mu(I) = f(x)$ a.e The understanding of the symbol $I \Rightarrow x$ (read "$I$ contracts to $x$") here is that $x$ is an endpoint of the interval $I$ and $\delta(I)$ (the diameter of $I$) tends to zero.

Thus Lebesgue's classic theorem can take the following form.

THEOREM. *If $f$ is summable on sets of finite measure and $\sigma$ is the indefinite integral of $f$, then*

$$(*) \qquad D\sigma(x) = \lim_{I\Rightarrow x}\frac{\sigma(I)}{\mu(I)} = f(x) \quad \text{a.e.}$$

2.2 **The general case of** $R_n$, $n \geqq 1$. We wish to generalize the theorem of 2.1 to Euclidean $n$-dimensional space $R_n$. For simplicity, we take the case $n = 2$ and note that everything we say in the two dimensional setting has an analogue in $n$-dimensional space $R_n$. Where such an analogue is not obvious, we shall state it. Otherwise, we shall state all our definitions and results for $R_2$ and leave it to the reader to make the obvious modifications for $R_n$, $n > 2$.

In order to generalize the theorem stated in Section 2.1, we want to do two things. First, we wish to determine which families of sets $\mathscr{I}$ are suitable for playing the role of the intervals, and then we wish to agree on what meaning we should give to the symbol $I \Rightarrow x$. More precisely, we wish to decide what it means to say that a sequence of sets in $\mathscr{I}$ contracts to a point of $R_2$. Once these two questions are decided, the idea, as in $R_1$, is to take a function $f$, summable on the sets of $\mathscr{I}$, take its average over $I \in \mathscr{I}$, and let $I$ contract to $x$. We would like this limit, $\lim_{I\Rightarrow x}\sigma(I)/\mu(I)$, to exist, where $\sigma(I) = \int_I f\,d\mu$, and to equal $f$ a.e. When the limit exists at a point $x$ (regardless of how $I \Rightarrow x$) we shall call it the **derivative** of $\sigma$ at $x$ (with respect to $\mu$ and relative to the differentiation basis $(\mathscr{I}, \Rightarrow)$) and denote it by $D\sigma(x)$. Whether or not the derivative exists at a point $x$, we can define the **upper** and **lower derivatives** by

$$\bar{D}\sigma(x) = \sup \limsup_{I \Rightarrow x} \frac{\sigma(I)}{\mu(I)}, \qquad \underline{D}\sigma(x) = \inf \liminf_{I \Rightarrow x} \frac{\sigma(I)}{\mu(I)},$$

where sup and inf are taken over all sequences of sets in $\mathscr{I}$ contracting to $x$. When confusion arises because more than one basis is under consideration, we shall indicate the basis in our notation. In certain cases, it will be desirable to modify our notation slightly, but no confusion should arise.

Let us now return to the two items we mentioned at the beginning of the last paragraph. First, there are several natural choices for the family $\mathscr{I}$ with respect to which we wish to differentiate. For example, we might let $\mathscr{I}$ consist of all squares (including interior) or of all disks, or of all two dimensional intervals (i.e., rectangles with sides parallel to the coordinate axes), or of *all* rectangles, or of any of a number of other families of sets. Once we have a family $\mathscr{I}$, we must decide what we mean by the expression $I \Rightarrow x$. Again there are a number of possibilities. For example, we might agree that $I \Rightarrow x$ means

$$\text{(i)} \quad x \in I \quad \text{and} \quad \mu(I) \to 0,$$

or

$$\text{(ii)} \quad x \in I \quad \text{and} \quad \delta(I) \to 0,$$

or we might modify the requirement that $x \in I$ be requiring only that $x$ be in the closure of $I$, or by requiring additionally that $x$ be in a particular position in $I$. Any of these notions generalizes the one dimensional case. A bit of reflection shows that if we want the analogue of the theorem of Section 2.1 to hold, the first alternative must be discarded at least for many natural choices for $\mathscr{I}$. For example, if $\mathscr{I}$ denotes the family of intervals of $R_2$, and if $f$ denotes the characteristic function of the upper half plane, then $\bar{D}\sigma \equiv 1$ while $\underline{D}\sigma \equiv 0$, so $D\sigma$ exists nowhere. So, for the time being, we shall select alternative (ii). We shall consider other possibilities later.

That being the case, the situation is as follows. If we take $\mathscr{I}$ to be the family of disks or squares (in which case differentiation relative to $(\mathscr{I}, \Rightarrow)$ is often called **ordinary differentiation**), then (*) (see the theorem of Section 2.1) holds for all locally summable $f$ [95], [137], [144]. If we take $\mathscr{I}$ to be the family of two dimensional intervals (in which case differentiation relative to $(\mathscr{I}, \Rightarrow)$ is called **strong differentiation,** then (*) holds for all bounded summable $f$, (and some other functions), but not for all summable functions [95], [144]. Finally, if we take the family $\mathscr{I}$ of *all* rectangles, then (*) does not even hold for all bounded summable $f$. In fact, (*) does not even hold for all characteristic functions of open sets [21]. What is it that causes these differences? This question can be answered at various levels. To understand the differences fully, one must understand the proofs and counterexamples required to justify the statements. But, short of going into the necessary details, let us try to give some sort of indication of what is involved. There are two classical theorems which are of fundamental importance in differentiation theory; the LEBESGUE DENSITY THEOREM and the VITALI COVERING THEOREM. We state these theorems for the family of squares. Let $\mu^*$ denote Lebesgue outer measure.

LEBESGUE DENSITY THEOREM. *Let $A \subset R_2$. For almost every $x \in A$, $x$ is a point of density of $A$; that is, whenever $I \Rightarrow x$, then*

$$\lim_{I \Rightarrow x} \frac{\mu^*(I \cap A)}{\mu(I)} = 1 .$$

DEFINITION. Let $A \subset R_2$. If $\mathscr{I}^*$ is a family of squares such that for every $x \in A$ there exists a sequence $\{I_k(x)\}$ of squares in $\mathscr{I}^*$ such that $I_k(x) \Rightarrow x$, then we say $\mathscr{I}^*$ is a **Vitali cover** of $A$.

VITALI COVERING THEORFM. *If $\mathscr{I}^*$ is a Vitali Cover of a set $A$, and $\varepsilon > 0$, then there exists a sequence $I_1, I_2, \cdots$, chosen from $\mathscr{I}^*$, such that*

(i) $\mu(A \sim \bigcup I_k) = 0$,

(ii) $I_m \cap I_n = \varnothing$ *if* $m \neq n$,

(iii) $\mu(\bigcup I_k \sim \bar{A}) < \varepsilon$, *where $\bar{A}$ is a measurable cover for $A$.*

The definition of Vitali cover, as well as the two theorems, were given relative to the differentiation basis consisting of the squares. Making obvious changes, we arrive at the definition of a Vitali cover relative to any basis $(\mathscr{I}, \Rightarrow)$. If the Density Theorem holds for $(\mathscr{I}, \Rightarrow)$, we say $(\mathscr{I}, \Rightarrow)$ has the **density property.** If the Vitali theorem holds for $(\mathscr{I}, \Rightarrow)$ we say $(\mathscr{I}, \Rightarrow)$ has the **strong Vitali property.** It turns out that if $\mathscr{I}_1$ consists of the squares, $\mathscr{I}_2$ of the disks, $\mathscr{I}_3$ of the intervals, and $\mathscr{I}_4$ of the rectangles, then $\mathscr{I}_1$, $\mathscr{I}_2$ and $\mathscr{I}_3$ have the density property, but $\mathscr{I}_4$ doesn't, while $\mathscr{I}_1$ and $\mathscr{I}_2$ have the strong Vitali property, but $\mathscr{I}_3$ and $\mathscr{I}_4$ don't.

We summarize with a chart.

| $\mathscr{I}$ | * holds for | density and/or Vitali property possessed by $\mathscr{I}$ |
|---|---|---|
| $\mathscr{I}_1$ squares | all $f \in L_1$ | both |
| $\mathscr{I}_2$ disks | all $f \in L_1$ | both |
| $\mathscr{I}_3$ intervals | all $f \in L_\infty$ | density property only |
| $\mathscr{I}_4$ rectangles | not even all characteristic functions of open sets | neither |

The chart tells the story in part. If $(\mathscr{I}, \Rightarrow)$ possesses the strong Vitali property, then (*) holds for all $f \in L_1$. The density property for $(\mathscr{I}, \Rightarrow)$ is necessary and sufficient for (*) to hold for all $f \in L_\infty$.

In connection with the family $\mathscr{I}_4$, we mention an example due to Nikodym [106]. Nikodym constructed a closed set $S \subset R_2$ having positive measure such that to almost every point $x \in S$, there corresponds a line segment $L_x$ whose intersection with $S$ is exactly $x$. It is easy to verify that for each such $x$, there exists a sequence $\{R_k\}$ of rectangles contracting to $x$, each containing a segment of $L_x$, such that $\lim_{k\to\infty}\mu(S \cap R_k)/\mu(R_k) = 0$. Thus, $\mathscr{I}_4$ does not possess the density property.

Other proofs that $\mathscr{I}_4$ does not have the density property can be found in Busemann and Feller [21] and Papoules [109].

We shall return to the question, "What causes (*) to hold for some bases $(\mathscr{I}, \Rightarrow)$ but not for others?," but we wish to ask this question in a somewhat broader context, and it seems desirable to digress for the moment with three comments. After these digressions, the last of which is rather lengthy, we shall return to the question at hand.

COMMENT 1. We mentioned that (*) holds for a class of functions somewhat larger than the class $L_\infty$ when $\mathscr{I} = \mathscr{I}_3$. Zygmund showed in 1934 [169] that this class includes all functions in $L_p$ for every $p > 1$. Then, in 1936, Jessen, Marcinkiewicz, and Zygmund [71] extended this result to the class $L\log L$ of functions $f$ such that $|f|\log^+|f| \in L_1$, where $\log^+ x = \max(0, \log x)$. The class $L\log L$ contains every $L_p$ class, $p > 1$. Thus, the class for which (*) holds relative to $\mathscr{I}_3$ is in some sense large. In another sense, however, this class is very small, thought of as a subset of $L_1$. Specifically, Saks [141] showed that the class of functions whose integrals are strongly differentiable at even one point is a first category subset of $L_1$. (See also [95].) What Saks actually proved is that there exists a residual subset $S$ of $L_1$ such that if $f \in S$, then $\bar{D}_s\sigma \equiv +\infty$, where $\bar{D}_s\sigma$ denotes strong differentiation (i.e., differentiation with respect to $\mathscr{I}_3$). Thus while each absolutely continuous Lebesgue-Stieltjes measure is differentiable a.e. relative to $\mathscr{I}_1$ or $\mathscr{I}_2$, "most" such measures are nowhere differentiable relative to $\mathscr{I}_3$.

COMMENT 2. The density property can be cast in a Vitali form by weakening the conclusion of the strong Vitali property to allow "arbitrarily small" overlap of the elements of $\mathscr{I}$ appearing in the conclusion of the Vitali theorem. This can be done in a number of ways (see Section 3.1). For example, de Possel [126] proved the following theorem (which holds in abstract spaces as well, see Section 6).

THEOREM. *A necessary and sufficient condition that $(\mathscr{I}, \Rightarrow)$ possess the density property is that $(\mathscr{I}, \Rightarrow)$ possess the* ***weak Vitali property***, i.e., *the strong Vitali property with conditions* (ii) *and* (iii) *replaced by*

$$\Sigma\, \mu(I_k) < \frac{1}{\alpha}\mu^*A,$$

*where $\alpha$ is any preassigned positive number less than* 1.

We note that this condition simultaneously guarantees that both the "total overlap" of the sets $I_k$ and the "overflow" of $\cup I_k$ over $A$ will be "arbitrarily small".

COMMENT 3. If one studies Banach's proof [6] of the Vitali covering theorem, one sees that essential to the proof is a simple fact about squares in $R_2$. If $S_1$ and $S_2$ are any concentric squares with sides parallel to the coordinate axes such that $\delta(S_2) \geqq 9\,\delta(S_1)$, and $x_1 \in S_1$, $x_2 \in \sim S_2$, then the distance between $x_1$ and $x_2$ is at least twice the diameter of $S_1$. (Similar statements are valid for cubes in $R_n$, the number 9 being replaced by the number $3^n$). The important thing is that there exists one number which works for all pairs of concentric squares. It is easy to verify that the corresponding statement for intervals in $R_2$ does not hold—no single constant works uniformly for all pairs of concentric intervals.

For purposes of generalization, it is more convenient to state this property in a somewhat different form. Let $S$ be any square whose sides are parallel to the coordinate axes. There is a number (9) such that if $H$ denotes the union of all squares which are no larger than $S$ and which intersect $S$, then $\mu(H)/\mu(S) \leqq 9$. The letter $H$ is used to suggest the term "halo" because of the appearance of the set $H$ (actually of the set $H \sim S$). It turns out that the basis $(\mathscr{I}, \Rightarrow)$ possesses the strong Vitali property if it possesses a certain halo condition, suggested by, but more general than, the halo property of squares mentioned above.

DEFINITION. A family of sets $\mathscr{I}$ is said to possess the **Morse halo property** provided there is a bounded positive function $\Delta$ defined on $\mathscr{I}$ such that the "$\Delta$-halo" defined by

$$H_\Delta(I) = \bigcup \{J \in \mathscr{I} : I \cap J \neq \varnothing,\ \Delta(J) \leqq 2\Delta(I)\}$$

satisfies the inequality

$$\mu^*(H_\Delta(I)) \leqq \lambda\mu(I)$$

for some $\lambda < \infty$ and all $I \in \mathscr{I}$.

THEOREM. *Suppose $(\mathscr{I}, \Rightarrow)$ possesses the Morse halo property, where $\mathscr{I}$ consists of bounded closed sets and $I \Rightarrow x$ means $x \in I$ and $\delta(I) \to 0$. Then $(\mathscr{I}, \Rightarrow)$ possesses the strong Vitali property.*

This theorem was first proved by Morse [97] for metric spaces. He assumed the members of $\mathscr{I}$ to be closed. One can, however, verify (see Alfsen [1]) that the result holds if for every finite collection $I_1, I_2, \cdots, I_n \in \mathscr{I}$, and $x \in \sim \cup I_k$, there exists $I \in \mathscr{I}$ such that $x \in I$ and $I \cap I_k = \varnothing$ if $k = 1, \cdots, n$. One can't quite have the strong Vitali property if one doesn't put some sort of restriction on $\mathscr{I}$. For example, if one adds a collection of rational points near $I$ to each $I \in \mathscr{I}$ in an appropriate manner, one can't possibly get the sets $\{I_k\}$ to be disjoint.

We note that the function $\Delta$ might measure the diameter of the sets in $\mathscr{I}$, or their Lebesgue measure, but these are not the only possibilities.

It turns out that the Lebesgue density property is equivalent to a different sort of halo condition under certain circumstances.

Let $\mathscr{I}$ be the family of sets of some differentiation basis. Let $\alpha \in (0,1)$ and let $\mu(A) < \infty$. Let $A_\alpha$ denote the union of all $I \in \mathscr{I}$ for which $\mu(I \cap A) > \alpha\mu(I)$.

THEOREM. *Let $\mathscr{I}$ be a family of bounded open sets in $R_2$ and suppose $\mathscr{I}$ is closed under homothetic transformations. Let $I \Rightarrow x$ mean $x \in I$ and $\delta(I) \to 0$. Then $(\mathscr{I}, \Rightarrow)$ possesses the density property if and only if $(\mathscr{I}, \Rightarrow)$ possesses the* ***weak halo property****, namely, for each $\alpha \in (0,1)$ and each $A$ of finite measure, the set $A_\alpha$ has finite measure.*

This theorem is due to Busemann and Feller [21].

We have seen that the strong Vitali property and the weak Vitali property (equivalently, the density property) are related to certain halo conditions.

Let us now pose our question in a somewhat broader context. Given a family $F$ of functions, under what conditions imposed on a differentiation basis $(\mathscr{I}, \Rightarrow)$ does (*) hold for each $f \in F$? As we already saw, these questions can be answered for certain classes $F$ in terms of various types of Vitali, density, or halo properties. In order to obtain a more complete answer to this question, we consider several other Vitali and halo properties.

DEFINITION [21]. Let $\mathscr{I}$ be the family of sets of a differentiation basis. Let $S \subset R_2$, $0 < \alpha < 1$, and $\beta > 0$. Define $\sigma_{\alpha\beta}(S)$ by

$$\sigma_{\alpha\beta}(S) = \bigcup \{I \in \mathscr{I} : \mu(I \cap S) > \alpha\mu(I),\ \delta(I) < \beta\}.$$

DEFINITION. The family $\mathscr{I}$ is said to possess the **halo evanescence property,** if for each decreasing sequence $\{S_n\}$ of bounded measurable sets whose intersection is empty, and each decreasing sequence $\{\beta_k\} \downarrow 0$ of real numbers, $\mu(\sigma_{\alpha\beta_n}(S_n)) \to 0$ for all $\alpha$.

THEOREM [21]. *Let $(\mathscr{I}, \Rightarrow)$ be a differentiation basis where $I \Rightarrow x$ means that $x \in I$ and $\delta(I) \to 0$. Suppose each $I \in \mathscr{I}$ is a bounded open set. Then $(\mathscr{I}, \Rightarrow)$ possesses the density property if and only if $(\mathscr{I}, \Rightarrow)$ possesses the halo evanescence property.*

We observe that we do not require $\mathscr{I}$ to be closed under homothetic transformations here as we did in the weak halo property stated above.

Note also, that in the Morse halo the "nucleus" consists of members of $\mathscr{I}$, while in the Busemann-Feller halo the nucleus consists of arbitrary bounded measurable sets. In either case, the rest of the halo consists of unions of sets in $\mathscr{I}$. We shall define one more halo [21], and observe that the nucleus here consists of the union of a finite number of measurable sets.

DEFINITION. Let $\mathscr{I}$ be a family of sets in $R_2$. Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be arbitrary positive numbers, and let $S_1, \cdots, S_n$ be arbitrary bounded measurable sets. Then

$$H(\alpha_1, \cdots, \alpha_n : S_1, \cdots, S_n) = \bigcup \left\{ I \in \mathscr{I} : \sum_{k=1}^{n} \alpha_k \mu(I_n \cap S_k) > \mu(I) \right\}.$$

DEFINITION. The family $\mathscr{I}$ possesses the **strong-(Busemann-Feller) halo property** provided for each collection $(\alpha_1, \cdots, \alpha_n;\ S_1, \cdots, S_n)$,

$$\mu(H(\alpha_1, \cdots, \alpha_n : S_1, \cdots, S_n)) < C \sum_{k=1}^{n} a_k \mu(S_k),$$

where the constant $C$ depends only on $\mathscr{I}$.

THEOREM. *Let $\mathscr{I}$ be a family of bounded open sets in $R_2$, closed under homothetic transformations. Let $I \Rightarrow x$ mean $x \in I$ and $\delta(I) \to 0$. Then (*) holds for every $f \in L_1$ if and only if $\mathscr{I}$ possesses the strong halo property.*

We state one more Vitali property of "intermediate" strength [64].

DEFINITION. Let $(\mathscr{I}, \Rightarrow)$ be a differentiation basis in $R_2$ and let $p \geqq 1$. We say $(\mathscr{I}, \Rightarrow)$ possesses the ***p*-Vitali property** if for each set $A \subset R_2$ and $\varepsilon > 0$, and each Vitali cover $\mathscr{I}^*$ of $A$, there exists a sequence $\{I_k\}$ of sets in $\mathscr{I}^*$ such that

(i) $\mu(A \sim \bigcup I_k) = 0$;

(ii) $\int [\varepsilon_w(x)]^p d\mu < \varepsilon$, where $\varepsilon_A(x)$ denotes one less than the number of sets in $\{I_k\}$ to which $x$ belongs, and the integral is taken over $\bigcup I_k$;

and

(iii) $\mu^*(\bigcup I_k \sim \bar{A}) < \varepsilon$, where $\bar{A}$ is a measurable cover for $A$.

Condition (ii) guarantees a small overlap, condition (iii) a small overflow.

The two theorems stated below indicate the role a $p$-Vitali property plays in differentiation theory. Both of these theorems can be found in [64]. They apply in abstract settings as well as in our present concrete setting.

THEOREM. *If $p > 1$ and $(\mathscr{I}, \Rightarrow)$ possesses the p-Vitali property, then (*) holds for every $f \in L_q$, with $1/p + 1/q = 1$.*

THEOREM. *If (*) holds relative to $(\mathscr{I}, \Rightarrow)$ for every $f \in L_q$, $(1/p + 1/q = 1)$, then $\mathscr{I}$ possesses the p'-Vitali property for every p' satisfying $1 \leqq p' < p$.*

We note that these two theorems are not quite converses of each other. We do not know whether the direct converse of the first of these theorems is valid.

Putting together a few of the preceding results, we have the following theorem:

THEOREM. *Let $\mathscr{I}$ be a family of bounded open sets and let $I \Rightarrow x$ mean $x \in I$ and $\delta(I) \to 0$. The following conditions are equivalent:*

1. *The Fundamental Theorem of Calculus holds for all $f \in L_\infty$.*
2. *$(\mathscr{I}, \Rightarrow)$ possesses the weak Vitali property.*
3. *$(\mathscr{I}, \Rightarrow)$ possesses the Density property.*

4. $(\mathscr{I}, \Rightarrow)$ *possesses the Halo Evanescence property.*
*If, in addition,* $\mathscr{I}$ *is closed under homothetic transformations, then condition* 5 *below is equivalent to the others.*
5. $(\mathscr{I}, \Rightarrow)$ *possesses the weak halo property.*

The theorem gives an indication of when a basis $(\mathscr{I}, \Rightarrow)$ is suitable for differentiating all integrals of the entire class $L_\infty$. If, for example, $(\mathscr{I}, \Rightarrow)$ has the weak Vitali property, it differentiates all integrals of $L_\infty$ functions. But it might differentiate integrals of certain additional functions as well, as one can see from the theorem of Jessen-Marcinkiewicz and Zygmund. Given an $f$, not necessarily in $L_\infty$, we ask for conditions under which $(\mathscr{I}, \Rightarrow)$ differentiates $\sigma = \int f\,d\mu$. The following theorem [64] gives an answer to this question.

THEOREM. *If* $(\mathscr{I}, \Rightarrow)$ *possesses the weak Vitali property,* $f \in L_1$, *and* $\sigma = \int f\,d\mu$, *then* $(\mathscr{I}, \Rightarrow)$ *differentiates* $\sigma$ *to* $f$ *if and only if* $(\mathscr{I}, \Rightarrow)$ *possesses the weak Vitali property with respect to* $\sigma$.

This theorem generalizes to abstract spaces.

Table I on page 10 summarizes our results and provides a partial answer to the question raised on page 7. By $I \Rightarrow x$ we mean $x \in I$ and $\delta(I) \to 0$.

As we mentioned earlier, all of the results of this section have analogues in $R_n$, $n > 2$. Many have analogues in abstract measure spaces and we shall discuss some of these in Chapter 6. The one result whose $n$ dimensional analogue is not obvious (to state) is the result of Jessen-Marcinkiewicz and Zygmund. We now state the $n$-dimensional version [71].

THEOREM. *If* $f(\log^+|f|)^{n-1}$ *is summable, then the integral of* $f$ *is strongly differentiable to* $f$ *a.e.*

As before, strong differentiability of $\sigma$ means $\sigma$ is differentiable with respect to the intervals ($n$-dimensional rectangular parallelepipeds with faces parallel to coordinate hyperplanes).

This result is the best possible in a certain sense. Let $\phi$ be an increasing function defined on $[0, \infty)$ such that $\phi(0) = 0$, $\phi(t) > 0$ for $t > 0$, and such that $\liminf_{t\to\infty} \phi(t)/t > 0$. Let $L_\phi$ denote the class of functions $f$ such that $\phi(|f|)$ is summable on the unit $n$-dimensional cube $S$. If for each $f \in L_\phi$ the integral of $f$ is almost everywhere strongly differentiable, then $f(\log^+|f|)^{m-1}$ is summable on $S$. Thus for $\phi_0(t) = t(\log^+ t)^{m-1}$ in particular, $L_{\phi_0}$ is the largest Orlicz class with the property that the integral of every one of its members is strongly differentiable a.e.

A related result can be found in Saks [142].

Recently, Zygmund [171] obtained an extension in a different direction. Suppose $s$ is an integer satisfying $1 \leqq s \leqq n$. Then (*) holds, with respect to the family of all $n$-dimensional intervals with at most $s$ different edge lengths, for all $f$ such that $|f|(\log^+|f|)^{s-1}$ is summable. For $s = n$, this is just the Jessen-Marcinkiewicz-

TABLE I

| $F$ | Vitali property on $(\mathscr{I}, \Rightarrow)$ | halo property on $(\mathscr{I}, \Rightarrow)$ |
|---|---|---|
| $L_1$ | strong Vitali suffices for (*). | (1) Morse halo property suffices for (*) if $\mathscr{I}$ consists of bounded closed sets.<br>(2) Busemann-Feller strong halo property is necessary and sufficient for (*) if $\mathscr{I}$ consists of bounded open sets and is closed under homothetic transformations. |
| $L_{q'}$ for all $q' \in (q, \infty]$ | $p'$-Vitali for all $p' \in [1, p)$ is necessary and sufficient for (*). $(1/p + 1/q = 1)$. | (*) (*) |
| $L_\infty$ | Weak Vitali is necessary and sufficient for (*). Equivalently, density property is necessary and sufficient for (*). | (1) Halo evanescence property is necessary and sufficient for (*) if $\mathscr{I}$ consists of bounded open sets.<br>(2) Weak halo property is necessary and sufficient for (*) if $\mathscr{I}$ consists of bounded open sets and is closed under homothetic transformations. |

(*) (*) A halo property on $(\mathscr{I}, \Rightarrow)$ which is sufficient for the differentiation of integrals of functions in $L_q$ can be found in Hayes [59].

Zygmund theorem, while for $s = 1$ this is the Fundamental Theorem of Calculus for the cubes.

Another extension of the theorem can be found in Smith [151]. See also the end of Section 6.2 below.

We mention in closing this section that most books dealing with the differentiation of integrals use Vitali-type theorems and density theorems rather than halo conditions to obtain their results. A proof of the Jessen-Marcinkiewicz-Zygmund theorem in $R_2$, based on halo properties of the two dimensional intervals, can be found in Burkhill [20]. Halo properties relating to classes of functions other than $L \log L$ can be found in Hayes [60]. Additional papers dealing with differentiation of integrals in $R_n$ are [4], [35], and [162].

## III. COVERING THEOREMS AND DENSITY THEOREMS IN EUCLIDEAN SPACES

In Chapter 2 we saw one of the reasons that covering properties (and density properties) are important in the theory of differentiation; namely, the composition of the class of functions whose integrals differentiate back to the function is closely related to the type of covering property that the differentiation basis possesses. These properties have other uses as well, and it is often convenient to obtain variants of the covering theorem for specific purposes. We begin this chapter with a discussion of some of the variants possible in $R_n$. We then address ourselves to the notion of **perfect basis**, that is, a basis for which the strong Vitali property holds not only with respect to Lebesgue measure, but with respect to every other Lebesgue-Stieltjes measure as well. We end the chapter with a few remarks concerning density theorems.

3.1. **Variants of Vitali properties.** Let $(\mathscr{I}, \Rightarrow)$ denote a differentiation basis in $R_n$. Let $\{E_k\}$ be a sequence of bounded measurable sets. For each $k$, let $\alpha_k = \sup\{\mu(E_k)/\mu(I): I \in \mathscr{I},\ E_k \subset I\}$. The sequence $\{E_k\}$ is called **regular** (with respect to $\mathscr{I}$) provided there is a positive constant $\alpha$ such that $\alpha_k > \alpha$ for all $k$. Now suppose $(\mathscr{J}, \Rightarrow)$ is a differentiation basis for $R_n$, where $\mathscr{J}$ is a family of closed sets, $\Rightarrow$ has the usual meaning, and for each $x \in R_n$, every sequence $\{J_k\}$ of sets in $\mathscr{J}$ such that $J_k \Rightarrow x$ is regular.

THEOREM. *If $(\mathscr{I}, \Rightarrow)$ possesses the strong Vitali property, so does $(\mathscr{J}, \Rightarrow)$.*

Vitali's theorem is sometimes stated for such systems [144] with $\mathscr{I}$ denoting the family of cubes.

Another variant of Vitali's theorem has proved useful in connection with certain questions concerning the differentiation of integrals [71], [171], and with certain questions in multiple Fourier series [170], [71]. Let $\phi_1, \phi_2, \cdots, \phi_n$ be increasing functions defined on $(0, \infty)$ such that $\lim_{t\to 0}\phi_i(t) = 0$ for all $i$, and $\phi_i(t) > 0$ for all $t > 0$. Suppose $\mathscr{I}$ denotes the family of all $n$-dimensional intervals of the form $\{(x_1, x_2, \cdots, x_n): a_1 \leqq x_i \leqq b_i$ where $b_i - a_i = \phi_i(t)$ for some $t\}$. With the usual notion of contraction, $(\mathscr{I}, \Rightarrow)$ possesses the strong Vitali property. One can prove this statement directly, or by observing that $(\mathscr{I}, \Rightarrow)$ possesses the Morse halo property with $\Delta(I) = \mu(I)$. To verify the Morse halo property, we observe that since each of the functions $\phi_i$ is increasing, if $\Delta(J) \leqq 2\Delta(I)$ for $I, J \in \mathscr{I}$, then every edge of $J$ has length at most twice that of the corresponding edge of $I$. Thus $H_\Delta(I)$ is contained in the union of those $K \in \mathscr{I}$ intersecting $I$ and whose edge lengths are twice the corresponding ones of $I$. It is easy to verify that if $S$ is this union, then $\mu(H_\Delta(I)) \leqq \mu(S) < k_n\mu(I)$ where $k_n$ is a constant depending only on the dimension of the space.

We observe that the family $\mathscr{I}$ need not be regular with respect to the family of $n$-dimensional cubes. If, for example, $n = 2$, and $\phi_1$ has a vanishing derivative at

$t = 0$ while $\phi_2$ has an infinite derivative at $t = 0$, then the family $\mathscr{I}$ is not regular.

The weak, strong, and "intermediate" Vitali properties found in Chapter 2 differ in the amount of overlap allowed in the sets of the sequence selected to almost cover $I$. There are various other forms the covering properties might take. For example, variation is possible in the *manner* in which one measures overlap. Thus, one might define the overlap of $\{I_k\}$ to be

$$\sum \mu(I_k) - \mu(\bigcup I_k)$$

provided $\mu(\bigcup I_k) < \infty$. When $\mu(\bigcup I_k) < \infty$, this measure is the same as that stated in Chapter 2. The Vitali properties we have considered involve (possibly) infinite sequences of sets $\{I_k\}$. By requiring only that $\mu^*(\bigcup (I_k \sim A)) < \varepsilon$ one can replace the infinite sequence $\{I_k\}_{k=1}^{\infty}$ by a finite sequence.

For certain studies it is necessary to measure the overflow and overlap not with respect to Lebesgue measure $\mu$, but with respect to another measure $\sigma$. For example, if $\sigma = \int f d\mu$, a number of results can be stated in terms of Vitali properties that $(\mathscr{I}, \Rightarrow)$ may possess with respect to $\sigma$. A relatively complete discussion of the relationship between Vitali properties of various sorts and the ability of a differentiation basis to differentiate integrals and measures can be found in Hayes and Pauc [64]. The results obtained in that work apply to abstract measure spaces as well as to Euclidean spaces.

A number of other covering theorems have been used by various authors for special purposes. For example, covering theorems which have proved useful in dealing with problems in singular integrals have been advanced by several authors. See Guzman [48] for statements and applications of such theorems as well as for references to various other related applications. Some of these theorems apply to spaces more general than $R_n$.

3.2 **Perfect bases.** To this point we have been concerned almost entirely with Euclidean spaces furnished with Lebesgue measure and with bases $(\mathscr{I}, \Rightarrow)$ for which $I \Rightarrow x$ means $x \in I$ and $\delta(I) \to 0$. Most of the results we mentioned apply equally well to other measures and to certain other meanings of $\Rightarrow$. We shall now consider the following question: Under what circumstances does a basis $(\mathscr{I}, \Rightarrow)$ have the property that the strong Vitali property hold with respect to *every* Lebesgue-Stieltjes measure? We shall call such a basis a **perfect basis** and we shall see that the meaning of $\Rightarrow$ will, of necessity, be somewhat restricted.

The first to consider this question was Besicovitch [9], [10], who showed that if $\mathscr{I}$ denotes the family of spheres in $R_n$, and $I \Rightarrow x$ means $x$ is the *center* of $I$ and $\delta(I) \to 0$, then $(\mathscr{I}, \Rightarrow)$ is a perfect basis (see also Sikorski [150] and Iséki [67] for similar results in $R_1$). Besicovitch extended this result to bases which are $\sigma$-regular with respect to the spheres, and he was able to drop the requirement that $I \Rightarrow x$ means $x$ is the center of $I$. However, $\sigma$-regularity depends on the measure $\sigma$, so in this more general setting, Besicovitch's second result applies only to measures which satisfy the regularity condition relative to the given basis.

Consider, for example, the basis of closed disks in $R_2$, where $I \Rightarrow x$ means $x \in I$ and $\delta(I) \to 0$. Let $\bar{\sigma}$ be the completion of that measure $\sigma$, which for any Borel set $A$ satisfies $\sigma(A) = \mu(A \cap R_1)$, where $\mu$ is one dimensional Lebesgue measure. Let $\mathscr{I}^*$ consists of the disks tangent to an interval $[a,b]$ of $R_1$. Then $\bar{\sigma}([a,b]) = b-a$, but no denumerable subcollection of $\mathscr{I}^*$ covers $\bar{\sigma}$-almost all of $[a,b]$. Thus, $(\mathscr{I}, \Rightarrow)$ does not even possess the weak Vitali property.

So what does it take for $(\mathscr{I}, \Rightarrow)$ to be perfect? Because of Besicovitch's first result, one might feel that if the elements of $\mathscr{I}$ are "nearly" spherical, and if $I \Rightarrow x$ involves some notion which indicates that $x$ is "nearly" a center of $I$, then $(\mathscr{I}, \Rightarrow)$ would be perfect. We shall see that while this feeling is in part correct, care has to be taken to formulate the conditions of "near sphericalness" and "near center" accurately. In fact, the perfection of a basis is more closely related to starshapedness than to sphericalness.

The results we are about to discuss, as well as the remarks contained in the preceding paragraph, are due to Morse [98]. We begin with a bit of terminology.

DEFINITION. The **internal radius** of a set $B$ at $x$ is defined to be the supremum of the set $\{r : S(r,x) \subset B\}$, where $S(r,x)$ denotes the closed sphere with center $x$ and radius $r$. The **hub radius** of $B$ at $x$ is defined to be the internal radius of the convex kernel of $B$ at $x$.

Now let $(\mathscr{I}, \Rightarrow)$ be a differentiation basis for $R_n$. If $\mathscr{I}$ consists of closed sets, and $I_k \Rightarrow x$ means

$$\delta(I_k) \to 0 \quad \text{and} \quad \limsup_{k\to\infty} \frac{\delta(I_k)}{\text{hub radius of } I_k \text{ at } x} < \infty,$$

then $(\mathscr{I}, \Rightarrow)$ is called a **star basis.**

For example, if $\mathscr{I}$ denotes the family of closed spheres in $R_n$ and $I_k \Rightarrow x$ means $x$ is the center of $I_k$ and $\delta(I_k) \to 0$, then $(\mathscr{I}, \Rightarrow)$ is a star basis.

THEOREM. *Each star basis is perfect.*

Observe that the definition of a star basis imposes a condition on $\Rightarrow$ which indicates a sense in which $I_k \Rightarrow x$ requires $x$ to be a "near center" of $I_k$. If we weaken our requirement by requiring only that $\delta(I_k) \to 0$ and $\limsup_{k\to\infty} \delta(I_k)/(\text{internal radius of } I_k \text{ at } x) < \infty$, then the basis need not be perfect. In fact, Hayes and Morse [Ann. 13] have given an example of a basis $(\mathscr{I}, \Rightarrow)$ in $R_2$ whose members consist of closed starshaped sets, and such that for every $x \in R_2$, $\limsup_{k\to\infty}\delta(I_k)/(\text{internal radius of } I_k \text{ at } x) = 2$ whenever $I_k \Rightarrow x$, even though $(\mathscr{I}, \Rightarrow)$ is not perfect. The basis $(\mathscr{I}, \Rightarrow)$ is not even **universal**; that is, it is not the case that $(\mathscr{I}, \Rightarrow)$ differentiates the integral of every function summable with respect to some Lebesgue-Stieltjes measure. (Every perfect basis is universal because the appropriate entries in Table I, p. 10, apply to all Lebesgue-Stieltjes measures.) We note that in this example, if $I_k \Rightarrow x$, then $I_k$ contains one sphere about $x$ and is contained in another sphere about $x$ with the ratio of the radii of these spheres tending to 1 as $k$ tends to infinity. Yet the basis is not perfect. As Morse points out, the condition involving the hub radius requires less in the way of sphere shapedness but more in the way of starshapedness.

Star bases have certain advantages over other types of bases. These are discussed in [98, p. 432].

Related results, with applications to singular integral equations can be found in Guzman [48]. Goffman [47] has used Morse's theorem to obtain a characterization of a certain class of linearly continuous functions.

3.3 **Density theorems.** We turn now to a discussion of various sorts of density theorems. In Chapter 2 we considered the Lebesgue density property and pointed out that if $(\mathscr{I}, \Rightarrow)$ possesses this density property, then this basis differentiates integrals of bounded summable functions. The underlying measure was Lebesgue measure.

For certain applications, it is desirable to have analogous theorems with respect to other measures. An exhaustive study of the types of density theorems which have applications to the theory of surface area can be found in Mickle and Radó [90]. To develop a theory here would take us too far afield, so we shall content ourselves with a few sample results. We note that some of the theorems below are actually differentiation theorems (every density theorem can be interpreted in this way) but we include them in this section because the authors looked upon their results as density theorems.

Sierpinski [149] and Besicovitch [7] considered questions of linear density of planar sets of points. Let $\Gamma$ denote Carathéodory linear measure in $R_2$. Let $C(x,r)$ denote the disk having center $x$ and radius $r$. Let $A$ be a linearly measurable subset of $R_2$. We define the **upper** and **lower linear densities** of $A$ at $x$ by

$$D^*(x,A) = \limsup_{r\to 0} \frac{\Gamma(A \cap C(x,r))}{2r}$$

and

$$D_*(x,A) = \liminf_{r\to 0} \frac{\Gamma(A \cap C(x,r))}{2r}.$$

Three theorems of Besicovitch will suffice to illustrate the extent to which the Lebesgue density theorem holds.

THEOREM 1. *At $\Gamma$-almost all points $x$ of $A$*

$$\tfrac{1}{2} \leqq D^*(x,A) \leqq 1 \quad \textit{and} \quad 0 \leqq D_*(x,A) \leqq 1.$$

These limits are the best possible in the sense that there exist sets of $\Gamma$-measure greater than 0 for which $D^*(x,A) = \frac{1}{2}$ or $D^*(x,A) = 1$, or $D_*(x,A) = 0$ or $D_*(x,A) = 1$, as the case may be, at $\Gamma$-almost all points of $A$.

THEOREM 2. *At $\Gamma$-almost all points in the complement of $A$, $D^*(x,A)=D_*(x,A)$ $= 0$.*

THEOREM 3. *If $A \subset B$ and $\Gamma(A) > 0$, then at $\Gamma$-almost all points of $A$, $D^*(x, A) = D^*(x, B)$ and $D_*(x, A) = D_*(x, B)$.*

Theorem 3 generalizes the linear Lebesgue density theorem; for if $A \subset R_1$, we can take $B = R_1$.

Sierpinski obtained results analogous to Theorem 1 and 2 for sets $A$ which are not necessarily linearly measurable.

We note in these theorems that the denominators in the expressions defining the densities are "linear". Related theorems hold with the denominator representing the right dimension. Let $\Gamma$ be any Carathéodory outer measure, and let $\mu^*$ be Lebesgue outer measure in $R_n$. Let $\mathscr{I}$ be the family of $n$-dimensional closed spheres, where $I \Rightarrow x$ means $x$ is the center of $I$ and $\delta(I) \to 0$. Let $E$ be a $\mu$ measurable set. Then $\lim_{I \Rightarrow x}\Gamma(E \cap I)/\mu(I) = 0$ or $\infty$ for $\mu$-almost all points of $R_n \sim E$.

If $\psi$ is an outer measure (not necessarily a Carathéodory outer measure), one can only conclude that $\limsup_{I \Rightarrow x}\psi(E \cap I)/\mu(I) = 0$ or $\infty$ for $\mu$-almost all points in $R_n \sim E$. (One cannot replace the expression "lim sup" by "lim".) See Mickle-Radó [88] for proofs of these statements. Conditions under which this replacement is permissible can be found in [90]. A number of related results (many in the setting of metric measure spaces) can be found in this work. See also [89], [41], and [100].

## IV. APPLICATIONS

The theory of differentiation of integrals in Euclidean spaces, along with the tools of that theory (covering theorems, density theorems, etc.) have uses in a number of mathematical fields. In some cases the tools can be used directly to carry out the necessary estimates or calculations, and in other cases the problems at hand quickly reduce to questions concerning the differentiation of integrals or set functions. In the present section we give some indications of how the theory arises in certain mathematical areas and indicate some of the results obtainable. In order to avoid getting too far afield, we consider only those cases which can be described easily and which involve very little technical machinery to describe. In some cases we include some of the details of the treatment; in other cases, we do little more than indicate how certain notions can be defined or interpreted in terms of derivatives. Our setting is that of $R_n$, primarily for $n \geqq 2$. We note that several different differentiation bases appear in the examples.

4.1 **Equivalences of cross partial derivatives.** Let $\mathscr{I}$ denote the family of intervals in $R_2$ and let $\Rightarrow$ denote the usual notion of contraction. Let $f$ be summable on, say, the square $S = [0,1] \times [0,1]$, and let $F$ be defined on $S$ by $F(\xi,\eta) = \int_0^\xi \int_0^\eta f\,dx\,dy$. The function $F$ determines an interval function $\sigma$ in the usual way: if $I = [\xi, \xi + h] \times [\eta, \eta + k] \subset S$, we define $\sigma$ by $\sigma(I) = F(\xi + h, \eta + k) - F(\xi, \eta + k) - F(\xi + h, \eta) - F(\xi, \eta)$. Then

$$\frac{\sigma(I)}{\mu(I)} = \frac{1}{k}\left[\frac{F(\xi + h,\eta + k) - F(\xi,\eta + k)}{h} - \frac{F(\xi + h,\eta) - F(\xi,\eta)}{h}\right]$$

$$= \frac{1}{h}\left[\frac{F(\xi + h,\eta + k) - F(\xi + h,\eta)}{k} - \frac{F(\xi,\eta + k) - F(\xi,\eta)}{k}\right].$$

Now suppose the partial derivatives of $F$ exist in a neighborhood of $(\xi,\eta)$. If $\sigma$ is strongly differentiable at the point $(\xi,\eta)$, then this limit is just the double limit of the displayed expressions. The iterated limits, however, give the cross partials $\partial^2 F/\partial x\,\partial y$ and $\partial^2 F/\partial y\,\partial x$. It follows that if $\sigma$ is strongly differentiable a.e. in $S$, and $F$ possesses first partial derivatives in $S$, then

$$\lim_{I \Rightarrow (\xi,\eta)} \frac{\sigma(I)}{\mu(I)} = \frac{\partial^2 F}{\partial x\,\partial y}(\xi,\eta) = \frac{\partial^2 F}{\partial y\,\partial x}(\xi,\eta) = f(\xi,\eta)$$

for almost all points $(\xi,\eta) \in S$. In particular, the cross partial derivatives are equal almost everywhere in $S$.

Now, even if $F$ does not possess first partial derivatives everywhere in $S$, it is the case that $F$ possesses first partials a.e. in $S$. Thus, if $\sigma$ is strongly differentiable a.e. in $S$, we can still conclude the equality a.e. of the cross partials, provided we relax slightly the requirement that a mixed partial can be defined at a point only if the first partials exist in a neighborhood of that point.

Comparable results hold in $R_n$. See [21] for a complete development.

Now let $f$ be a measurable function in $R_1$. A theorem of Lusin [82] (see also Saks [142]) asserts that there exists a continuous function $F$ such that $F' = f$ a.e. Certain analogous results in $R_n$ have been obtained by Saks [142]. For example, if $f$ is a measurable function in $R_2$, there exists a continuous measure $\sigma$ such that the strong derivative of $\sigma$ equals $f$ a.e. In terms of point functions, the theorem takes the form that every measurable $f$ defined on $R_2$ is a.e. the cross partial derivative of some continuous function $F$. In the one dimensional case, if $f$ is summable on sets of finite measure, then clearly $F$ can be taken to be the integral of $f$. But in $R_2$, the integral of $f$ might be *nowhere* strongly differentiable. What functions $F$, then, have the desired property? Saks showed that if $f$ is summable on sets of finite measure, there always is an $F$ of the form $F = F_1 + F_2$, where $F_1$ is the integral of $f$ and $F_2$ is continuous and singular, such that $\partial^2 F/\partial x \partial y = \partial^2 F/\partial y \partial x = f$ a.e. Additional restrictions can be put on $F$. For example, one can require (even if $f$ is not summable!) that $F = F_1 + F_2$ where $F_1$ is the integral of some bounded function, and $F_2$ has continuous partial derivatives $\partial^k F_2/\partial x^k$ and $\partial^k F_2/\partial y^k$ of all orders everywhere, and cross partial derivatives $\partial^2 F_2/\partial x \partial y$ and $\partial^2 F_2/\partial y \partial x$ (equal to $f$) almost everywhere.

These results have analogues in spaces $R_n$ of higher dimension.

A number of additional results of a somewhat related nature can be found in Goffman [46] and Easton, Tucker, and Wayment [37].

4.2 **Multiple Fourier series.** A standard theorem in Fourier series asserts that if $f$ is in $L_1[0, 2\pi]$ and periodic with period $2\pi$, then the $(C,1)$ means of $f$ converge a.e. to $f$. We consider now some analogous questions for summable functions on $S \equiv [0, 2\pi) \times [0, 2\pi)$ (extended periodically, of period $2\pi$ in each variable, to all of $R_2$). Let $\sigma_{m,n}(f : x_0, y_0)$ denote the $mn$th Fejer mean of $f$ at $(x_0, y_0)$:

$$\sigma_{mn}(f : x_0, y_0) = \frac{1}{4mn\pi^2} \int_{x_0-\pi}^{x_0+\pi} \int_{y_0-\pi}^{y_0+\pi} f(x, y) \left( \frac{\sin m(x - x_0)/2}{\sin (x - x_0)/2} \right)^2 \left( \frac{\sin n(y - y_0)/2}{\sin (y - y_0)/2} \right)^2 dx dy.$$

Let $f$ be a positive function, of period $2\pi$ in each variable, which is summable on $S$ and for which the upper strong derivative of its integral is identically $+\infty$. Since the upper strong derivative is infinite everywhere, it can be computed at each point $p$, with $p$ the center of the intervals contracting to $p$. Then there exist two sequences of positive integers $\{m_k\}$ and $\{n_k\}$ such that $m_k \to \infty$, $n_k \to \infty$, and

$$\frac{m_k n_k}{4\pi^2} \int_{x_0 - 1/m_k}^{x_0 + 1/m_k} \int_{y_0 - 1/n_k}^{y_0 + 1/n_k} f(x, y) dx\, dy \to \infty .$$

One can verify that the corresponding Fejer means dominate the left side of this last expression and thus approach $+\infty$. Since the Fejer means are just the $(C, 1)$ partial sums of the double Fourier series, this shows that the $(C, 1)$ sums converge *nowhere* to $f$.

This result is due to Zygmund (see Saks [141], whose proof we reproduced above). If $f \in L_p$, $p > 1$, then the $(C, 1)$ means converge to $f$ a.e. [169]. The same is true for each $f \in L \log L$ [170], [71]. This last result is in a sense the best possible [71].

A number of results for multiple Fourier series which involve the differentiation of integrals can be found in Zygmund [170]. For example, if in the question considered above we require that $m$ and $n$ tend to infinity in such a way that $n/m$ and $m/n$ remain bounded, then the $(C, 1)$ means converge to $f$ a.e. Here we require only that $f \in L_1$. Comparable results hold in $R_n$, $n > 2$. A proof of this fact can be based on the variant of Vitali's theorem (involving the functions $\phi_i$) considered in Section 3.1. (See also [85].) Note the similarity between these results and certain analogous results concerning the differentiation of integrals. In Section 7.2 we shall indicate why this similarity is not so surprising as it may at first seem.

Another result of interest is that if $f$ is summable on $S \equiv [0,2\pi] \times [0,2\pi]$ and its integral is strongly differentiable, then its Fourier series is Abel summable ($A^*$) (see [170] for definition of $A^*$ summability) a.e. on $S$. If, in addition, the integral of $|f|$ is strongly differentiable, then the Fourier series of $f$ is $(C, 1)$ summable a.e. on $S$. We note that it is possible for the strong derivative of the integral of $f$ to exist a.e. while the strong derivative of the integral of $|f|$ exists nowhere [108].

4.3. **Boundary behavior of harmonic functions.** Let $u$ be a positive harmonic function defined in the unit disk $|z| < 1$. A standard theorem of Fatou's guarantees that $u$ has radial limits a.e. Specifically, $\lim_{r\to 1} u(re^{i\theta}) = u(\theta)$ exists for almost every $\theta$ between 0 and $2\pi$. A result which gives some sort of indication of the radial behavior at every $\theta$ can be formulated in terms of differentiation of measures.

We begin by recalling a few facts about representation of harmonic functions [137]. If $\sigma$ is a Borel measure on the circle $|z| = 1$, and $P_r(\theta - t)$ denotes the **Poisson kernel,**

$$P_r(\theta - t) = \operatorname{Re}\left[\frac{e^{it} + z}{e^{it} - z}\right] = \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2},$$

then the function

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\,d\sigma, \qquad 0 \leqq r < 1$$

is harmonic in the open unit disk $|z| < 1$. In fact, given a harmonic function $u$ such that

$$\sup\left\{\int_{-\pi}^{\pi} |u(re^{i\theta})|\,d\theta \,:\, 0 \leqq r < 1\right\} < \infty,$$

there exists a measure $\sigma$ such that the representation above is valid.

Let $\mathscr{I}$ be the family of arcs on the unit circle $|z| = 1$, and for fixed $\theta$ take $I \Rightarrow e^{i\theta}$ to mean that $e^{i\theta}$ is the center of the arc $I$ and $\delta(I) \to 0$. Let $\mu$ denote Lebesgue measure on $|z| = 1$, and let $\bar{D}\sigma$, $\underline{D}\sigma$, and $D\sigma$ denote the upper derivative, lower derivative, and derivative of $\sigma$ with respect to $\mu$ relative to the basis $(\mathscr{I}, \Rightarrow)$, i.e., the **symmetric derivative.**

THEOREM. *Under the conditions described above, for each $\theta$,*

$$\underline{D}\sigma(e^{i\theta}) \leqq \liminf_{r\to 1} u(re^{i\theta}) \leqq \limsup_{r\to 1} u(re^{i\theta}) \leqq \bar{D}\sigma(e^{i\theta}).$$

*In particular, at every point of differentiability $(\mathscr{I}, \Rightarrow)$ of $\sigma$, the radial limit $\lim_{r\to 1} u(re^{i\theta})$, exists and equals $D\sigma(e^{i\theta})$.*

Certain comparable results are valid for harmonic functions defined on a half plane and for harmonic functions of several variables. We mention part of one such result.

THEOREM. *Let $u$ be a positive harmonic function in the upper half space $\{(x\ y\ z): z > 0\}$. There exists a measure $\sigma$ defined on the Lebesgue measurable subsets of the plane $P = \{(x, y, z): z = 0\}$ such that if $(x_0, y_0, 0) \in P$ and the symmetric derivative of $\sigma$ with respect to Lebesgue measure exists at $(x_0, y_0, 0)$, then $\lim_{z \to 0} u(x_0, y_0, z)$ exists and equals that symmetric derivative evaluated at $(x_0, y_0, 0)$. In particular, $u$ possesses vertical limits at almost all points of $P$.*

The symmetric derivative as usual denotes

$$\lim_{I_\alpha \Rightarrow (x_0, y_0, 0)} \frac{\sigma(I_\alpha)}{\mu(I_\alpha)},$$

where $I_\alpha$ is the disk $\{(x, y, 0): x^2 + y^2 \leqq \alpha^2\}$ and $I_\alpha \Rightarrow (x_0, y_0, 0)$ means $\alpha \to 0$.

We mention that the validity of this theorem can be checked by carefully following the proof of Theorem 1, Section 5 of Carleson [22].

4.4 **Complex analysis.** Let $f$ be a continuous function of a complex variable defined on a domain $G \subset R_2$. For each interval $I \subset G$, let $[I]$ denote the boundary of $I$. Write

$$\int_{[I]} f(z)dz = \sigma_1(I) + i\sigma_2(I) = \sigma(I).$$

The function $\sigma_1$ and $\sigma_2$ are continuous additive interval functions, satisfying

$$|\sigma_k(I)| \leqq \left| \int_{[I]} f(z)dz \right| \leqq |\sigma_1(I)| + |\sigma_2(I)|,$$

for all $I$ and $k = 1, 2$. Now, even though $\sigma_1$ and $\sigma_2$ are not measures, these functions are defined over the sets of $\mathscr{I}$, the family of squares in $R_2$, so we can define the derivatives of $\sigma_1$ and $\sigma_2$ with respect to $\mu$ (relative to $\mathscr{I}$) in the obvious manner, where $I \Rightarrow z$ means $z \in I$ and $\delta(I) \to 0$. One can show then [144] (by using a bit more machinery concerning the differentiation of interval functions than we have developed), that if

$$\underline{D}|\sigma|(z) = \liminf_{I \Rightarrow z} \frac{|\sigma(I)|}{\mu(I)} = 0 \qquad \text{for almost all } z \in G$$

and

$$\bar{D}|\sigma|(z) = \limsup \frac{|\sigma(I)|}{\mu(I)} < \infty \qquad \text{except, perhaps, for a denumerable set,}$$

then $f$ is holomorphic in $G$.

Further theorems whose conclusions are that $f$ is holomorphic and whose proofs can be based on differentiation theory, can be found in Saks [144, p. 195 ff.].

Results analogous to those stated in Section 4.3 apply to bounded analytic functions defined on the open disk $|z| < 1$. Each such function $f$ is the Poisson integral of some complex measure $\sigma$ defined for the Borel subsets of $|z| = 1$, and the radial limit

$\lim_{r\to 1} f(re^{i\theta})$ exists *if and only if* the symmetric derivative of $\sigma$ with respect to Lebesgue measure $\mu$ exists at $e^{i\theta}$ [81]. In fact, the symmetric derivative in this case exists at $e^{i\theta}$ if and only if either one-sided derivative exists at $e^{i\theta}$ [12].

4.5 **Vector analysis.** Many of the terms appearing in what one might call vector analysis involve derivative of the type under consideration. We give a few examples.

Let $v$ be a continuous vector field defined in a neighborhood of a point $p_0$ in $R_3$. Let $C_n(p_0, r)$ denote the circle with center $p_0$ and radius $r$, lying in the plane through $p_0$ normal to $n$, oriented by the right-hand rule. We define the **upper circulation per unit area** of $v$ at $p_0$ in the direction $n$ by

$$D_n^* v(p_0) = \limsup \frac{1}{\pi r^2} \int_{C_n(p_0, r)} v \cdot t \, ds,$$

where $t$ denotes the unit tangent and $ds$ the differential of arc length. The **lower circulation per unit area** is then defined in the obvious way, and if the upper and lower circulations are equal and finite at $p_0$, their common value is designated by $D_n v(p_0)$ and called the **circulation per unit area** of $v$ at $p_0$ in the direction $n$.

It is easy to interpret this circulation as a derivative. Fix $n$. Let $\mathscr{I}$ denote the family of all disks $K_n(p_0, r)$ whose boundaries are of the form $C_n(p_0, r)$ as described above. Let $I \Rightarrow x$ have obvious meaning: $x$ is the center of $I$ and $\delta(I) \to 0$. Let $\sigma$ be defined by

$$\sigma(I) = \int_{C_n(p_0, r)} v \cdot t \, ds, \qquad \text{where } I = K_n(p_0, r).$$

Then $D_n v(p_0)$ is just the $(\mathscr{I}, \Rightarrow)$ derivative of $\sigma$ (with respect to two dimensional Lebesgue measure).

The curl of $v$ is then defined in terms of circulations.

The divergence of a vector field can also be interpreted in terms of derivatives. For example, in $R_2$, if $V(p) = [A(p), B(p)]$ is a continuous vector field in a neighborhood of $p_0$, we define the upper divergence by

$$\operatorname{div}^* V(p_0) = \limsup_{r\to 0} \frac{1}{\pi r^2} \int_{C(p_0, r)} A(p)dy - B(p)dx,$$

with a similar definition for $\operatorname{div}_* V(p_0)$ and the obvious definition for $\operatorname{div} V(p_0)$, when it exists. The interpretation of $\operatorname{div} V$ as a derivative $(\mathscr{I}, \Rightarrow)$ is similar to the analogous interpretation of the circulation as a derivative. (We could have used squares instead of disks as our differentiation basis.) We can also define an approximate divergence operator in terms of derivatives. Some interesting results concerning the curl, divergence and approximate divergence can be found in Shapiro [145], [146], and [147]. For interpretations of the gradient as a derivative, see Pauc [118] and [120].

4.6. **Surface area.** Just as differentiation theory plays an important role in the theory of arc length, so does it play an important role in the theory of surface area. To give a simple example of how differentiation arises in connection with surface area, we give a formula for the area of the surface corresponding to a continuous function $F$ defined on an interval $I_0 \subset R_2$. The notation we use is essentially that found in Saks [144], where all the necessary details can be found. See also [139], [140].

Let $F$ be continuous on $I_0 = [a_1, b_1] \times [a_2, b_2]$. Define $G_1$, $G_2$, and $G$, the expression of de Geöcze, by

$$G_1(F;I) = \int_{a_1}^{b_1} |F(x,b_2) - F(x,a_2)|\,dx,$$

$$G_2(F;I) = \int_{a_2}^{b_2} |F(b_1,y) - F(a_1,y)|\,dy,$$

and

$$G(F;I) = \{[G_1(F;I)]^2 + [G_2(F;I)]^2 + [\mu(I)]^2\}^{\frac{1}{2}},$$

where $\mu$ denotes two dimensional Lebesgue measure, and $I$ is an arbitrary interval in $I_0$. Define $\sigma$ by $\sigma(J) = \int_J G(F;I)$, the integral being a Burkhill integral. It turns out that $\sigma(J)$ is the surface area of the graph of $F$ over $J$. Let $(\mathscr{I}, \Rightarrow)$ be the differentiation basis consisting of the squares with the usual notion of contraction. Then

$$D\sigma(\xi,\eta) = \left\{\left[\frac{\partial F}{\partial x}(\xi,\eta)\right]^2 + \left[\frac{\partial F}{\partial y}(\xi,\eta)\right]^2 + 1\right\}^{\frac{1}{2}},$$

for almost all points $(\xi,\eta) \in I_0$. Thus, in order that the surface area be given by the elementary formula

$$\sigma(I) = \int_J\!\int \left\{\left[\frac{\partial F}{\partial x}\right]^2 + \left[\frac{\partial F}{\partial y}\right]^2 + 1\right\}^{\frac{1}{2}} d\mu = \int_J\!\int D\sigma(\xi,\eta)\,d\mu$$

for all intervals $J \subset I_0$, it is necessary and sufficient that $\sigma$ be absolutely continuous with respect to $\mu$. The absolute continuity of the interval function $\sigma$ is equivalent to the function $F$ being absolutely continuous in the sense of Tonelli.

4.7 **Change of variables in integration.** Certain formulae for change of variables (or measures) in multiple integrals can be expressed in terms of derivatives with respect to the cubes.

Suppose first that $\sigma$ is an absolutely continuous measure defined on the (Lebesgue) measurable subsets of $R_n$. Let us write $d\sigma/d\mu$ for the derivative of $\sigma$ with respect to $\mu$ relative to the basis of cubes with the usual notion of contraction For any $f$. for which $\int_E f\,d\sigma$ exists for a measurable set $E$, we have

$$\int_E f\,d\sigma = \int_E f\frac{d\sigma}{d\mu}d\mu.$$

This change of measure formula is valid also in abstract measure spaces (see Chapter 6) provided the Radon-Nikodym derivative can be represented as a derivative relative to some differentiation basis $(\mathscr{I}, \Rightarrow)$. In Section 6.4 below, we shall learn that every complete sigma-finite measure space has such a basis.

Now let $T$ be a mapping from an open set $V$ in $R_n$ to $R_n$, and let $A$ be a linear mapping of $R_n$ into itself. If for each $\varepsilon > 0$ there exists a $\delta > 0$ such that the inequality $\| T(x+h) - T(x) - Ah \| \leqq \varepsilon \| h \|$ holds for all $h \in R_n$ such that $\| h \| < \delta$, we say that $T$ is differentiable at $x$ with derivative $A$. The symbol $\| \ \|$ designates the norm $\| x \| = \max_{1 \leqq i \leqq n} |x_i|$, where $x = (x_1, x_2, \cdots, x_n)$. If $T$ is differentiable for every $x$ in $V$, we say $T$ is differentiable on $V$.

A standard change of variable formula for integrals of functions defined on open subsets of $R_n$ takes the following form [137].

THEOREM. *Let $T$ be differentiable on an open set $V \subset R_n$. Suppose*
(i) *$W = T(V)$ is a bounded open subset of $R_n$,*
(ii) *$T$ is one-to-one and $T^{-1}$ is continuous. Then, for each $f$ summable on $W$,*

$$\int_W f(y)dy = \int_V f(T(x))\,|J_T(x)|\,dx,$$

*where $J_T(x)$ is the Jacobian of $T$ at $x$.*

Now, it is not difficult to prove that if $\mathscr{I}$ is the family of open cubes in $R_n$ and $I \Rightarrow x$ means $x \in I$ and $\delta(I) \to 0$, then, for all $x$,

$$|J_T(x)| = \lim_{I \Rightarrow x} \frac{\mu(T(I))}{\mu(I)} = D\sigma(x),$$

where $\sigma(E) = \mu(T(E))$ for all measurable sets $E$. If we once again write $d\sigma/d\mu$ for $D\sigma$, we see that the change of variables formula can be written

$$\int_{T(V)} f\,d\mu = \int_V fT\frac{d\sigma}{d\mu}d\mu.$$

We note that the conditions we placed on $T$ are somewhat more restrictive than necessary. Several theorems far more delicate than ours can be found in Radó and Reichelderfer [131: 363–365]. In addition, that text contains a number of results which indicate the ways in which Jacobians and generalized Jacobians can be interpreted as derivatives of the type we consider in this article. We shall not include the details, which involve a number of notions and definitions which would take us too far beyond our present purposes.

4.8. **Mean value integrals.** Sometimes a statement about real functions can be proved easily and by elementary techniques, provided the functions under consideration are sufficiently smooth. In certain cases, the same statement can be proved about functions not satisfying the smoothness conditions by suitably approximating the functions by sufficiently smooth functions for which the statement has already been proved. In this section we briefly discuss one such approximation method which has proved to be useful in a number of applications, and which involves the differentiation of integrals.

Let $f$ be summable on compact subsets of $R_2$. For each positive integer $n$, define a function $f_n$ by

$$f_n(x, y) = n^2 \int_{I_n(x,y)} f\, d\mu,$$

where $I_n(x, y)$ is the square $\{(\xi, \eta): x \leqq \xi \leqq x + 1/n,\ y \leqq \eta \leqq y + 1/n\}$. The function $f_n$ is called the **mean value integral** of $f$ of **index** $n$. If $\sigma$ is the integral of $f$, $\sigma(E) = \int_E f\, d\mu$ for all bounded measurable sets $E$, then it is clear that

$$\frac{\sigma(I_n(x, y))}{\mu(I_n(x, y))} = f_n(x, y)$$

for all $n$ and all $(x, y) \in R_2$. Since the basis $(\mathscr{I}, \Rightarrow)$ of squares, where $I \Rightarrow (x, y)$ means $(x, y)$ is the lower left corner of the square $I$ and $\delta(I) \to 0$, possesses the strong Vitali property, it follows that $f_n \to f$ a.e.

Mean value integrals have uses in a number of fields. For example, for applications to surface area theory see Cesari [24], Saks [144], Bray [13], Radó [129], [130], and Morrey [96]; for applications to potential theory, see Evans [39], [40], and Riesz [135]; for applications to transformations $T$ from $R_2$ to $R_2$, see Radó and Reichelderfer [131]. We mention that in some applications it is more convenient to require the squares to be centered at the point or to replace the squares with disks.

We state one sample theorem of Radó [144] to give an example of the type of approximation theorem we mentioned in the introduction to Section 4.8.

RADÓ'S THEOREM. *Let $f$ be a continuous function defined on $R_2$ and let $\{f_n\}$ be the sequence of integral means determined by $f$. Then, for each interval $I_0 \subset R_2$, $S(f; I_0) = \lim_{n\to\infty} S(f_n; I_0)$, where, for example, $S(f; I_0)$ denotes the area of the surface given by $z = f(x, y)$ over the interval $I_0$.*

Now each of the functions $f_n$ has continuous partial derivatives (since $f$ is continuous), so the surface area formula works for each $f_n$. Thus, Radó's theorem tells us in particular that the surface area corresponding to a surface of the form $z = f(x, y)$, with $f$ continuous, is given as the limit of a sequence of surface areas of functions whose areas can be computed by the standard formula.

## V. MISCELLANEOUS RESULTS

5.1. **Generalizations of theorems on derivates in $R_1$.** Certain classical theorems involving differentiations of functions of one real variable have analogues in the higher dimensional setting. We begin with a few examples of this sort.

It is a well-known fact that the derivative $f'$ of a differentiable function $f$ defined on $R_1$ has the **Darboux** or **intermediate value property**: if $f'(x_1) = \alpha$, $f'(x_2) = \beta$, and $\gamma$ is between $\alpha$ and $\beta$, then there exists a point $x$ between $x_1$ and $x_2$ such that $f'(x) = \gamma$. This result generalizes to derivatives of integrals, or, more generally, to derivatives of additive interval functions.

THEOREM (Misik [91]). *Let $(\mathscr{I}, \Rightarrow)$ denote the differentiation basis of cubes or of intervals in $R_n$ with the usual notion of contraction. Let $\sigma$ be a Lebesgue-Stieltjes measure, or, more generally, an additive interval function, which is differentiable $(\mathscr{I}, \Rightarrow)$ everywhere in $R_n$. For each $I \in \mathscr{I}$ there exists a point $x$ in the interior of $I$ such that $D\sigma(x) = \sigma(I)/\mu(I)$. Furthermore, for $x_1 \in I$, $x_2 \in I$ and $\gamma$ between $D\sigma(x_1)$ and $D\sigma(x_2)$, there exists $x_3 \in I$ such that $D\sigma(x_3) = \gamma$.*

See also Neugebauer [104], Kametani [73], and Ridder [133] for related results.

S. Marcus posed certain questions concerning Darboux properties of Jacobians and hyperbolic derivatives. Since the former can sometimes be interpreted as derivatives with respect to cubes and the latter are just strong derivatives, Misik's results give partial answers to the problem posed by Marcus.

In addition to having the Darboux property, every derivative is also in the **first class of Baire**; that is, the limit of a sequence of continuous functions. In fact, every bounded derivative has a much more restrictive property called $M_4$ by Zahorski [168]. This property is too complicated to state here, but we mention that Misik [94] showed that the comparable property holds for derivatives of additive interval functions in $R_n$. The basis here can consist of cubes or of all intervals. In particular, such derivatives are in the first class of Baire.

One of the outstanding problems dealing with ordinary derivatives is the problem of characterizing derivatives in terms of metric, topological, or measure-theoretic language. A discussion of the problem can be found in [168], [17], [14]. Not every function in the first Baire class and possessing the Darboux property is the derivative (everywhere) of some other function. It is natural to ask whether there exists some property which distinguishes the class of derivatives from other members of the class of Darboux-Baire 1 functions. (The question is analogous to a comparable one for integrals. The feature of the class of integrals which distinguishes an integral from an arbitrary continuous function of bounded variation is that every integral satisfies Lusin's condition (N).) While no completely satisfying answer to the question has been given, Neugebauer [102] has answered the question in terms of properties of interval functions. An analogous solution for the comparable question stated for derivatives of measures (or at least for additive interval functions) has been obtained recently by Misik [93]. We shall not state Misik's result here.

Let $f$ be a function of a real variable, and let $\sigma$ be the associated interval function: $\sigma([a,b]) = f(b) - f(a)$. The **Dini derivatives** of $f$ can be defined in terms of $\sigma$. Let $(\mathscr{I}, \underset{1}{\Rightarrow})$ be the basis of closed intervals, where $I \underset{1}{\Rightarrow} x$ means $x$ is the left-hand end point of $I$ and $\delta(I) \to 0$. The Dini derivates $D^+f$ and $D_+f$ are just the upper and lower $(\mathscr{I}, \Rightarrow)$ derivates of $\sigma$. A similar interpretation exists for left Dini derivates, giving rise to a differentiation basis $(\mathscr{I}, \underset{2}{\Rightarrow})$. Two theorems comparing Dini derivatives (of opposite sides) are due to Young [165] and Neugebauer [103].

YOUNG'S THEOREM. *For any function $f$, the upper derivate on one side is less than the lower derivate on the other side except, perhaps, for a denumerable set.*

NEUGEBAUER'S THEOREM. *If $f$ is continuous, then the two upper derivates are equal except for a set of the first category. The same is true of the two lower derivates.*

In our language, these two theorems compare the derivates of $\sigma$ with respect to the two bases $(\mathscr{I}, \underset{1}{\Rightarrow})$ and $(\mathscr{I}, \underset{2}{\Rightarrow})$. Both of these theorems admit of generalization [15], a (slightly simplified) version of which we now state.

Let $\mathscr{I}$ be the family of sets homothetic to a fixed bounded open set $I_0$ in $R_n$. Fix two positions on the boundary of $I_0$. For $I \in \mathscr{I}$, write $x \underset{1}{\in} I$ $(x \underset{2}{\in} I)$ if $x$ is the point on the boundary of $I$ corresponding to the first (resp., second) position. Let $I \underset{1}{\Rightarrow} x$ $(I \underset{2}{\Rightarrow} x)$ mean that $x$ is in the first (resp., second) position on the boundary of $I$ and $\delta(I) \to 0$.

THEOREM. *Let $(\mathscr{I}, \underset{1}{\Rightarrow})$ and $(\mathscr{I}, \underset{2}{\Rightarrow})$ be as above, and let $\sigma = \int f\,d\mu$. If both bases possess the weak Vitali property and both derivates equal $f$ a.e., then*

(a) *The upper $(\mathscr{I}, \underset{1}{\Rightarrow})$ derivate of $\sigma$ is no less than the lower $(\mathscr{I}, \underset{2}{\Rightarrow})$ derivate except on a set which contains at most a denumerable number of pairwise disjoint non-degenerate continua.*

(b) *The two upper derivates are equal except on a set of the first category.*

We caution that the term **non-degenerate continua** has the usual meaning in $R_2$ but an unusual meaning in $R_n$, $n \neq 2$. We shall not elaborate on this meaning.

The conditions placed on $(\mathscr{I}, \Rightarrow)$ and $\sigma$ in the preparation for this theorem were more restrictive than was necessary.

The theorem above also generalizes a related theorem concerning symmetric derivates found in [16].

Part (b) of the theorem is valid if one considers two different differentiation bases satisfying certain conditions. In particular, one can take $\mathscr{I}$ to be the squares and $\mathscr{J}$

to be the intervals in $R_2$. Thus, one can show that the set where the ordinary and strong upper derivates of an integral are unequal is a set of the first category. In view of Saks' result (Chapter 2) that for "most" $L_1$ functions $f$ the strong upper derivate of the integral $\sigma$ of $f$ is identically $+\infty$, and in view of the classical result that the ordinary derivate of $\sigma$ equals $f$ a.e., we see that for "most" $f$ in $L_1$ the or dinary upper derivate of $\sigma$ equals $f$ on a set of the first category and of full measure, and equals $+\infty$ on a residual null set.

An interesting structure theorem is due to Besicovitch [8].

THEOREM. *Let $f$ be summable in a domain $D \subset R_2$. Let $\sigma$ be the integral of $f$. Let $\bar{D}\sigma$ and $\underline{D}\sigma$ denote the strong upper and lower derivates of $\sigma$. For almost all $x \in D$, there are only four possibilities with respect to the differentiation of $\sigma$:*

(i) $\bar{D}\sigma(x) = \underline{D}\sigma(x) = f(x)$,

(ii) $\bar{D}\sigma(x) = +\infty$, $\underline{D}\sigma(x) = f(x)$,

(iii) $\bar{D}\sigma(x) = f(x)$, $\underline{D}\sigma(x) = -\infty$,

*and*

(iv) $\bar{D}\sigma(x) = +\infty$, $\underline{D}\sigma(x) = -\infty$.

This theorem is reminiscent of the Denjoy-Young-Saks theorem for Dini's derivates of functions of a real variable.

5.2 **Approximate continuity.** Let $f$ be defined in $R_n$. A point $x_0$ is said to be a **point of approximate continuity** of $f$ relative to $(\mathscr{I}, \Rightarrow)$ provided for each $\varepsilon > 0$, the set $\{x : |f(x) - f(x_0)| < \varepsilon\}$ has $x_0$ as a point of density. The bounded approximately continuous functions can be used to characterize **regular derivatives** $(\mathscr{I}, \Rightarrow)$, that is, derivatives computed with respect to differentiation bases $(\mathscr{I}, \Rightarrow)$ that are regular with respect to $\mathscr{I}$ (see Chapter 3).

THEOREM [136]. *Let $f$ be a bounded summable function in $R_n$. Then $f$ is approximately continuous $(\mathscr{I}, \Rightarrow)$ if and only if $f$ is the derivative of its integral with respect to every basis which is regular with respect to $(\mathscr{I}, \Rightarrow)$.*

This theorem is true pointwise; that is, $f$ is approximately continuous at $x_0$ if and only if $f$ is the derivative of its integral at $x_0$ with respect to every sequence $\{E_k\} \Rightarrow x$, which is regular.

We note that the boundedness of $f$ in the statement of the theorem cannot be dropped.

The theorem is valid without assuming any Vitali property for $(\mathscr{I}, \Rightarrow)$. If, for example, $\mathscr{I}$ denotes the family of all rectangles (sides not necessarily parallel to the coordinate axes) in $R_2$, the theorem is still valid. Thus, as we mentioned in Chapter 2, there is a characteristic function $f$ of a closed set $K$ of positive measure such that the derivative of the integral of the characteristic function of $K$ equals $f$ almost nowhere on $K$. The pointwise version of the theorem still guarantees that the $(\mathscr{I}_4, \Rightarrow)$ derivative of the integral will equal $f$ at every point of approximate continuity.

5.3. **Differentiation of interval functions.** Most of the preceding material dealt with the differentiation of integrals. A good deal of this material carries over to the differentiation of measures and of (countably) additive set functions. This follows readily from the Lebesgue decomposition theorem and the fact that singular measures have vanishing $(\mathscr{I}, \Rightarrow)$ derivatives a.e. if $(\mathscr{I}, \Rightarrow)$ differentiates integrals. In order for our notion of differentiation to have meaning, however, it is not necessary that $\sigma$ be defined with respect to any sets other than those of the differentiation basis. Nor is it really necessary that $\sigma$ be additive, although a number of theorems depend on the additivity of $\sigma$.

Let, for the moment, $(\mathscr{I}, \Rightarrow)$ be the family of intervals in $R_n$ with the usual meaning of contraction. If $\sigma$ is an additive interval function of bounded variation, much of the preceding material applies. A development of the theory in this setting can be found in Saks [144, Chapter 4]. If $\sigma$ is not assumed to be of bounded variation, the situation is somewhat more complicated. A. J. Ward [160], [161] obtained results for arbitrary interval functions which extend the results of Besicovitch mentioned in Section 5.1 above. Proofs of these results can also be found in [144; p. 133 ff], and an extension can be found in Saks [143].

A real valued function $\sigma$ defined on the family of $n$-dimensional intervals is called **additive** if $\sigma(I_1 \cup I_2) = \sigma(I_1) + \sigma(I_2)$ whenever $I_1 \cup I_2$ is an interval and $I_1$ and $I_2$ don't overlap (i.e., have no interior points in common).

THEOREM. *If $\sigma$ is an additive interval function in $R_n$, then*

(a) *$\sigma$ is differentiable with respect to the cubes at almost all points at which either extreme derivate is finite.*

(b) *$\sigma$ is differentiable with respect to the intervals (strongly differentiable) at almost all points at which both extreme strong derivates are finite.*

The differentiation of arbitrary functions defined on the sets of certain types of differentiation bases has been considered by Wright and Snyder [164]. These authors obtained necessary and sufficient conditions for certain types of derivates to be finite a.e., to be bounded a.e., and to be summable.

5.4. **Special differentiation bases.** We have already encountered certain special differentiation bases which appear frequently in the differentiation of integrals, measures, or interval functions: the families of spheres, cubes, and intervals. We shall consider another special type of basis called a net structure in Section 6.3. Certain other bases have desirable properties. For example, the star bases mentioned in Section 3.2 have the property [98] that if $T$ is a continuously differentiable one-to-one mapping of $R_n$ into itself, with nonvanishing Jacobian, then $T$ transforms any star basis into a star basis. Certain other special types of bases have been studied by Hayes [58] and Hayes and Morse [61], [62]. For example, in [61] a condition on a basis $(\mathscr{I}, \Rightarrow)$ for $R_n$ is given which will guarantee that if $\phi$ is a Lebesgue-Stieltjes measure in $R_n$, there exists a sub-basis $(\mathscr{J}, \Rightarrow)$ of $(\mathscr{I}, \Rightarrow)$ such that the strong

Vitali theorem holds for $(\mathscr{I}, \Rightarrow)$ with respect to $\phi$. (Compare with the perfect bases discussed in Section 3.2.)

5.5. **Extensions to infinite dimensional spaces.** We shall now consider an extension of the developments of the previous sections to a natural infinite dimensional analogue of $R_n$, or, more precisely, to its half open unit cube.

Let $S$ be an infinite set and let $[0,1)^S$ denote the family of functions from $S$ to $[0,1)$ furnished with the product topology. Let $s_1, s_2, \cdots, s_n$ be a finite subset of $S$ and let $a_1, a_2, \cdots, a_n$, $b_1, b_2, \cdots, b_n$ be numbers such that $0 \leqq a_i < b_i < 1$. The set $I = \{x \in [0,1)^S : a_1 \leqq x(s_1) \leqq b_1, \cdots, a_n \leqq x(b_n) \leqq b_1\}$ is called a **closed interval.** Similar definitions give rise to the **open** or **halfopen intervals**. Define a measure $\mu$ on the intervals by $\mu(I) = \prod_{i=1}^n (b_i - a_i)$. The interval function $\mu$ can then be extended to a measure on the sigma algebra generated by the intervals. Let $\mathscr{I}$ denote the family of all intervals and let $I \Rightarrow x$ mean that $x \in I$ and for each $s \in S$, the length of the $s$th coordinate of $I$ converges to 0.

It turns out [69], [70], [34] that the basis $(\mathscr{I}, \Rightarrow)$ is not suitable for differentiation even if $S$ is the set of positive integers, because $(\mathscr{I}, \Rightarrow)$ does not possess the density property. By considering a suitable sub-family of $\mathscr{I}$, however; one can obtain a basis $(\mathscr{I}, \Rightarrow)$ for which the strong Vitali property holds [144], [38].

We shall now construct such a basis. For each positive integer $m$, let $\mathscr{I}_m$ be the finite set of intervals whose $i$th component interval for $i \leqq m$ is of the form $[k_i/2^m, (k_i + 1)/2^m)$, where $k_i$ is an arbitrary positive integer less than $2^m$. The $i$th component interval for $i > m$ is the entire interval $[0,1)$. There are $2^{m^2}$ such intervals. For each $x \in [0,1)^S$ and each $m$, there is exactly one interval in the class $\mathscr{I}_m$ containing $x$. Let $\mathscr{I} = \bigcup_{m=1}^{\infty} \mathscr{I}_m$ and let $I_k \Rightarrow x$ mean that $I_k$ is the element of $\mathscr{I}_k$ containing $x$. The basis $(\mathscr{I}, \Rightarrow)$ forms what is called a **net structure** (see Section 6.3) and possesses the strong Vitali property. One can use this fact (along with the fact that the Fundamental Theorem of Calculus holds for differentiation with respect to nets) to prove an interesting theorem of Fubini type for the space $[0,1)^S$. A statement and proof of this theorem can be found in Saks [144; p. 157 ff]. The space $[0,1)^S$ has applications in probability theory.

## VI. DIFFERENTIATION OF INTEGRALS IN ABSTRACT MEASURE SPACES

The first to have studied the differentiation of integrals in abstract measure spaces appear to be Feller [42] and de Possel [125] [126]. These early works, particularly de Possel's, have been the starting point of a number of investigations by many authors. Among the major subsequent contributors to the development of the subject are Morse, Denjoy, Pauc, Hayes, Haupt, and Trjitzinsky. See in particular the works of these authors quoted in the introduction.

In his elegant and easy-to-read article, de Possel [126] furnished the notion of a differentiation basis $(\mathscr{I}, \Rightarrow)$ in abstract spaces. He then considered certain Vitali type properties and density properties that a basis might possess and proved the equivalence of five conditions on $(\mathscr{I}, \Rightarrow)$ including a weak Vitali property, the density property, and the Fundamental Theorem of Calculus for integrals of $L_\infty$ functions relative to $(\mathscr{I}, \Rightarrow)$. The other authors mentioned above obtained more delicate results, including also various halo properties. Whereas de Possel's article [126] is easy to read, that statement cannot be made of many of the other works on the subject. This is true partly because of the massive amount of technical machinery necessary to obtain certain desirable results (see for example the very general theory developed by Kenyon and Morse [75]), and partly because some of these authors had their own individual styles and made little use of the works of their predecessors. For example, in [31] and [154], [158] one finds no mention of de Possel's work.

In Section 6.1 we indicate the basic approach to differentiation theory in the abstract setting along with indications of certain modifications useful for special purposes. In Section 6.2 we discuss types of Vitali, density, and halo properties which, as in the Euclidean spaces, are necessary, or sufficient, or both, for a suitable theory of differentiation. We then consider two very simple types of bases in Section 6.3, and, in Section 6.4, consider the question of existence of bases which are suitable for differentiation. Section 6.5 contains a few miscellaneous remarks and results.

6.1. **Differentiation bases.** Let $(X, \mathscr{M}, \mu)$ be a measure space with a complete measure $\mu$. For our purposes it is generally convenient, (though not necessary [76]) to assume that the space is totally $\sigma$-finite, and we shall make that assumption throughout Chapter 6 unless we explicitly state otherwise. We wish to impose a differentiation basis $(\mathscr{I}, \Rightarrow)$ on $(X, \mathscr{M}, \mu)$ as we did in Chapter 2. To do so, we seek a family $\mathscr{I}$ of sets in $\mathscr{M}$ and a notion $\Rightarrow$ of contraction of sets of $\mathscr{I}$ to points of $X$ which are suitable for purposes of differentiation. We see immediately that certain problems exist in our selection of $\Rightarrow$ in the abstract setting that did not exist in the concrete setting. In $R_n$, we (generally) took $I \Rightarrow x$ to mean that $x \in I \in \mathscr{I}$ and $\delta(I) \to 0$. In the abstract setting, the notion of diameter is not available to us. We could, of course, say $I \Rightarrow x$ means that $x \in I \in \mathscr{I}$ and $\mu(I) \to 0$. This is a possibility, but it is somewhat too restrictive. Different authors have solved this problem in different ways, and we shall discuss some of these ways shortly. But first we wish to observe that while sequential contraction of sets in $\mathscr{I}$ to points of $X$ was generally sufficient in the Euclidean setting, it is often necessary in the abstract setting to deal with a more general notion of contraction, say, in the Moore-Smith sense.

In order not to commit ourselves at the start, we shall, with de Possel and others, take an axiomatic view towards $\Rightarrow$. Let $\mathscr{I}$ be a family of sets of positive measure, and let $\Rightarrow$ be a notion of contraction of certain generalized sequences of sets in $\mathscr{I}$ to points of $x$ such that the two conditions below are met:

(i) If $x \in X$, there exists at least one (generalized) sequence (in the sense of Moore-Smith) of sets of $\mathscr{I}$ which contract to $x$: in symbols $I \Rightarrow x$ or $I_\alpha \Rightarrow x$.

(ii) Every (cofinal) subsequence of a sequence contracting to $x$ also contracts to $x$.

If $\mathscr{I}$ is a family of sets of positive measure and $\Rightarrow$ satisfies (i) and (ii), then $(\mathscr{I}, \Rightarrow)$ is called a **differentiation basis** for $(X, \mathscr{M}, \mu)$. This is obviously a very general notion of differentiation basis, but for many purposes it is entirely appropriate for differentiation. We do not assume, incidentally, that if $I_\alpha \Rightarrow x$, then $x \in I_\alpha$ for all (or any!) $\alpha$. Nor do we assume that the meaning of $I \Rightarrow x$ is in any way related to the meaning of $I \Rightarrow y$ if $x \neq y$. In a sense, each $x \in X$ has contracting sequences of sets in $\mathscr{I}$ attached to it.

Now let $\sigma$ be any real valued function defined on the members of $\mathscr{I}$, and define the **upper derivate** of $\sigma$ with respect to $\mu$ at a point $x \in X$ by

$$\bar{D}\sigma(x) = \sup\{\limsup(\sigma(I_\alpha(x))/\mu(I_\alpha(x)))\},$$

where the expression in brackets denotes the limit superior for any one sequence $I_\alpha$ contracting to $x$, and the supremum is taken over all such sequences. The **lower derivate** $\underline{D}\sigma(x)$ is defined analogously and, when these two derivates are finite and equal at $x$, we denote their common value by $D\sigma(x)$ and call this value the **derivative** of $\sigma$ at $x$ (with respect to $\mu$ and relative to $(\mathscr{I}, \Rightarrow)$).

A number of authors have used the approach outlined above, in some cases with minor modification, to obtain a number of differentiation theorems analogous to those given in Chapter 2.

For some purposes it is desirable to be more specific in the meaning given to $\Rightarrow$. Some authors have required contractions to be tied in with a parameter which plays the role of diameter in $R_n$. Thus, suppose $\Delta$ is a positive finite function on $\mathscr{I} \subset \mathscr{M}$. We write $I \Rightarrow x$ provided $x \in I \in \mathscr{I}$ and $\Delta(I) \to 0$. Under certain circumstances [64] the suitability of a differentiation basis of this type can be described in terms of an analogue of the halo evanescence condition considered in Chapter 2.

Another approach to contraction which a number of authors have considered is the following. For $x \in X$, let $\mathscr{I}_x$ be the members of $\mathscr{I}$ containing $x$. Suppose $\mathscr{I}_x$ can be directed by downward inclusion. We define $I_\alpha \Rightarrow x$ to mean $I_\alpha \in \mathscr{I}_x$ for each $\alpha$, and the generalized sequence $I_\alpha$ is cofinal with $\mathscr{I}_x$. We shall use the phrase **contraction by inclusion** to describe this type of contraction. This notion of contraction by inclusion has several advantages. For one thing, as we shall see in Section 6.4 below, every complete totally or finite measure space has a differentiation basis $(\mathscr{I}, \Rightarrow)$ of this type, for which the strong Vitali theorem holds. Another advantage to contraction by inclusion is that if $(X, \mathscr{M}, \mu)$ is topologized by $\mathscr{T}$ and all members of $\mathscr{T}$ are measurable and of positive finite measure, then, by taking $\mathscr{I} = \mathscr{T}$, one has a natural notion of contraction tied in with the topology.

In a sense, however, this notion of contraction is not sufficiently general in that it does not include all the classical cases. For example, if $\mathscr{I}$ consists of the closed intervals in $R_2$ with the usual notion of contraction, then this is not a special case of contraction by inclusion (unless we assume $I \Rightarrow x$ implies $x$ is an interior point of $I$).

Various other special notions of $\Rightarrow$ are possible, but we shall not consider any more such notions at this time. We mention that one often puts restrictions on the family $\mathscr{I}$ when certain special situations are involved (see Section 6.3). Actually, for $(\mathscr{I}, \Rightarrow)$ to be suitable for differentiation, certain restrictions (in terms of Vitali, density, or halo conditions) on the pair $(\mathscr{I}, \Rightarrow)$ are necessary.

6.2. **Vitali, density, and halo conditions.** In Chapter 2 we saw the intimate tie-up between the ability of a differentiation basis to differentiate integrals and the types or strengths of Vitali, density, and halo properties the basis possessed. In the abstract setting there are analogues to all of the theorems implied by the chart found near the end of Chapter 2. The Vitali properties can be stated in manners entirely analogous to those stated in Chapter 2. No specific assumptions about the basis are necessary. Thus, no confusion should arise when we use terms such as "strong Vitali property".

The same is true of the density property. Halo properties generally require certain special restrictions on the measure spaces as well as on the individual sets in $\mathscr{I}$. Under proper circumstances, analogues for the halo theorems in Chapter 2 are possible. We do not go into the details here, but instead refer the reader to [64] where he can find an exhaustive study of various halo properties. In this same work, one can also find a number of Vitali properties of varying strengths and with a number of ways of measuring overlap. We shall not spell out the details, which are similar to those in $R_n$. We shall, however, indicate four approaches to covering theorems, three of which differ from those mentioned in Chapter 2. Each of these is motivated by part of Banach's proof [6] of the Vitali covering theorem in $R_2$. (See also [144].) Note that Morse's halo property is also related to an idea found in Banach's proof.

(A) Perhaps the most direct generalization of the Euclidean notion of a Vitali covering is the following: Let $(\mathscr{I}, \Rightarrow)$ be a differentiation basis for a measure space $(X, \mathscr{M}, \mu)$. A subset $\mathscr{J}$ of $\mathscr{I}$ is a Vitali covering of a set $A \subset X$ provided for each $x \in A$ there is a generalized sequence $\{J_\alpha\}$ of sets in $\mathscr{J}$ such that $J_\alpha \Rightarrow x$. This is essentially de Possel's approach and has been used, perhaps with modifications, by a number of authors. We consider this our basic notion of Vitali cover.

(B) A somewhat different approach, which has a number of virtues, is due to Alfsen [1]. Consider, for a moment, Banach's proof of the Vitali covering theorem [6].

Suppose $A$ is a set which is covered by (for simplicity) closed cubes in the Vitali sense. The construction of a disjoint sequence $\{I_k\}$ from the cover which almost covers $A$, involves an induction step. Suppose $I_1, I_2, \cdots, I_n$ are sets from the cover which are pairwise disjoint and cover part of $A$. One then uses the fact that the cover is a Vitali cover to observe that if $x \in A \sim \bigcup_{k=1}^{n} I_k$, there is an $I \in \mathscr{I}$ such that $x \in I$ and $I$ is disjoint from $\bigcup_{k=1}^{n} I_k$. It is easy to verify, more generally, that if $x \in A$ and $I_1, I_2, \cdots, I_n$ are sets of the cover, then either $x \in I_1 \cup I_2 \cup \cdots \cup I_n$ or there exists an $I$ in the cover which contains $x$ and is disjoint from each of the sets $I_1, I_2, \cdots, I_n$. Alfsen takes this notion as his "starting" point in defining a Vitali cover. Thus, let $(X, \mathscr{M}, \mu)$ be an arbitrary measure space. A collection $K$ of sets of positive finite measure is a Vitali cover of $A$, provided for $x \in A$ and any finite collection $K_1, K_2, \cdots, K_n$ of sets in $\mathscr{K}$, either $x \in K_1 \cup K_2 \cup \cdots \cup K_n$, or there exists $K \in \mathscr{K}$ such that $x \in K$ and $K$ is disjoint from $K_1 \cup K_2 \cup \cdots \cup K_n$.

Alfsen proves a number of Vitali type theorems, the first of which has an interesting corollary—a Morse halo condition.

THEOREM. *Let $\mathscr{K}$ be a Vitali covering of a set $A$ in a totally finite measure space and let $\Delta$ be a bounded positive function on $\mathscr{K}$. Define the Morse halo $H_\Delta(K)$ by*

$$H_\Delta(K) = \bigcup \{N : N \in \mathscr{K}, N \cap K \neq \varnothing, \Delta(N) \leqq 2\Delta(K)\}.$$

*If there exists a number $\lambda < \infty$ such that for all $K \in \mathscr{K}$, $\mu^*(H_\Delta(K)) \leqq \lambda\mu(K)$, then there exists a denumerable disjoint subcollection of $\mathscr{K}$ which covers almost all of $A$.*

We note that Alfsen's development does not explicitly employ a differentiation basis. Nevertheless, the Vitali type theorems he obtains can be used to check whether or not certain differentiation bases have the Vitali property. For example, he shows how the classical Vitali covering theorem, as well as a Vitali covering theorem for locally compact groups [25], follow easily from his results.

(C) A very general covering theorem of a somewhat different type has been established by Mickle and Radó [89]. Once again, motivation for this theorem can be found in Banach's proof of the Vitali covering theorem. Recall that Banach's proof, say, for the case of $R_2$ with $\mathscr{I}$ being the squares, involves an inductive selection of a sequence of squares $\{I_k\}$ which satisfy the conclusion of Vitali's theorem. The $n$ th square, $I_n$, is disjoint from $\bigcup_{k=1}^{n-1} I_k$ and has the property that among all squares in the cover which are disjoint from $\bigcup_{k=1}^{n-1} I_k$, none has diameter greater than twice that of $I_n$.

We shall state the Mickle-Radó theorem and indicate its relationship to the Vitali theorem. We mention that some of the abstract covering theorems of Morse [97] are special cases of this theorem.

Let $\mathscr{S}$ be a nonempty set (of any type). Let $\gamma$ and $\delta$ be two binary relations over $\mathscr{S}$. Write $S'\gamma S''$ $(S'\delta S'')$ to mean that the elements $S'$ and $S''$ of $\mathscr{S}$ satisfy the relation $\gamma$ ($\delta$ resp.). For $S \in \mathscr{S}$ define $N_\gamma(S) = \{S' : S' \in \mathscr{S}, S'\gamma S\}$, $N_\delta(S) = \{S' : S' \in \mathscr{S}, S'\delta S\}$, and $N(S) = N_\gamma(S) \cap N_\delta(S)$. For $\mathscr{E} \subset \mathscr{S}$, write $N_\gamma(\mathscr{E}) = \bigcup_{S \in \mathscr{E}} N_\gamma(S)$, $N_\delta(\mathscr{E}) = \bigcup_{S \in \mathscr{E}} N_\delta(S)$

and $N(\mathscr{E}) = \bigcup_{S \in \mathscr{E}} N(S)$, all these sets being empty if $\mathscr{E}$ is empty. Suppose

(i) $\gamma$ and $\delta$ are reflexive,

(ii) $\gamma$ is symmetric,

(iii) if $\mathscr{E} \subset \mathscr{S}$, $\mathscr{E} \neq \varnothing$, there exists $S \in \mathscr{E}$ such that $\mathscr{E} \subset N_\delta(S)$.

A subset $\mathscr{E}$ of $\mathscr{S}$ is called **scattered** if $\mathscr{E}$ contains no pair of distinct elements $S'$ and $S''$ such that $S'\gamma S''$.

THEOREM. *Under the circumstances described above, there exists a scattered subset $\mathscr{E}$ of $\mathscr{S}$ such that $\mathscr{S} = N(\mathscr{E})$.*

To interpret this theorem as a Vitali type theorem in $R_2$, let $\mathscr{S}$ be a family of squares covering a set $A$ in the Vitali sense. Let $S'\gamma S''$ mean $S' \cap S'' \neq \varnothing$, and let $S'\delta S''$ mean the diameter of $S'$ is no more than twice the diameter of $S''$. It is easy to verify that conditions (i), (ii), and (iii) above are satisfied. A scattered set with the present interpretation is just a disjoint family of squares from the Vitali cover $\mathscr{S}$. From the theorem, one can infer the existence of a disjoint collection $\mathscr{E}$ of squares in $\mathscr{S}$ such that if $S \in \mathscr{S}$, there exists $S' \in \mathscr{E}$ such that $S \cap S' \neq \varnothing$ and the diameter of $S$ is no greater than twice the diameter of $S'$. This result, by itself, is of course considerably weaker than the conclusion of Vitali's theorem. Nevertheless, it is useful in obtaining Vitali type theorems. The theorem does, however, apply under very general circumstances. See [89] for an application to metric spaces.

(D) Another approach to Vitali type theorems has been developed by Denjoy [31]. This approach generalizes the fact that the classical Vitali theorem applies to differentiation bases which are regular (see Section 3.1) with respect to a basis for which the Vitali theorem holds.

Denjoy considered two associated differentiation bases—to each element of one corresponds an element of the other, and vice-versa, with a regularity condition built in. His notion of contraction is in terms of the measure of the sets in the basis converging to 0, and he does not quite require that the point be in these sets. We shall not go into the details here. An outline of the results can be found in a series of papers [26], [27], [28], [29], [30], appearing in the *Comptes Rendus* of the French Academy of Science, a complete development (along with motivation) is contained in [31], and expository treatments of his development can be found in [32] and [33].

Another covering lemma, again based on Banach's proof of the Vitali covering theorem, has proved useful in applications.

LEMMA [152], [5]. *Let $\mathscr{S}$ be a family of spheres in a metric space. Let $S(x, r)$ denote the sphere with center $x$ and radius $r$. If* (i) *there is a number $R$ such that for every $S(x, r) \in \mathscr{S}$,*

$$0 < r < R,$$

*and* (ii) *for every disjoint sequence $\{S(x_n, r_n)]$ of spheres in $\mathscr{S}$,*

$$r_n \to 0,$$

*then there exists a disjoint sequence* $\{S(x_n, r_n)\}$ *of spheres in* $\mathcal{S}$ *such that* $\bigcup \{S : S \in \mathcal{S}\} \subset \bigcup S(x_n, 4r_n)$.

This lemma was used by Smith [152] to obtain generalizations of the Hardy-Littlewood inequalities for functions in $L^p$, $p > 1$, and $L \log L$. These inequalities, of course, have many applications in a number of branches of analysis, including the theory of the differentiation of integrals. For example, Smith [152] used his general inequality (for $L \log L$) to prove the generalization of the Jessen-Marcinkiewicz-Zygmund theorem stated below.

Other covering lemmas have been used for various purposes by a number of authors. See, for example, Aronszajn and Smith [5], Edwards and Hewitt [36], Guzman [48], Hörmander [66], Rauch [132], and Wiener [163].

Let $(X_1, \mathcal{M}_1, \mu_1)$ and $(X_2, \mathcal{M}_2, \mu_2)$ be two measure spaces furnished with differentiation basis $(\mathcal{I}_1, \underset{1}{\Rightarrow})$ and $(\mathcal{I}_2, \underset{2}{\Rightarrow})$. It is possible for each of the bases to possess the strong Vitali property without the same being true of the product basis in the product measure spaces. We saw an example of this in Chapter 2: the strong Vitali property is not possessed by the family $\mathcal{I}_3$ of two dimensional intervals with the usual notion of contraction. In $R_2$, however, the family $\mathcal{I}_3$ does possess the weak Vitali property, since that property is equivalent to the Lebesgue density property. Haupt and Pauc [53] have shown that this result holds in general: if $(\mathcal{I}_1, \underset{1}{\Rightarrow})$ and $(\mathcal{I}_2, \underset{2}{\Rightarrow})$ possess the weak Vitali property, then the same is true of the product basis $\mathcal{I}_1 \times \mathcal{I}_2$ in the product measure space.

Even though the strong Vitali property is not preserved under cartesian products, the theorem of Jessen-Marcinkiewicz-Zygmund concerning strong differentiation carries over to spaces more general than $R_2$. A theorem of this sort has been advanced by K. T. Smith [152].

This theorem states that if two metric spaces $M_1$ and $M_2$ are furnished with measures satisfying certain natural conditions (on relations between diameters and measures of spheres), and if $|f| \log^+ |f|$ is summable over $M_1 \times M_2$ furnished with the product measure, then the integral of $f$ is "strongly" differentiable a.e. to $f$. Here, as expected, "strong" differentiability is with respect to the basis $\mathcal{I} = \mathcal{I}_1 \times \mathcal{I}_2$, where $\mathcal{I}_1$ $(\mathcal{I}_2)$ consists of the closed spheres in $M_1$ ($M_2$ resp.), and contraction has the usual meaning.

6.3. **Net structures.** We turn now to a consideration of a special type of differentiation basis. Suppose $(X, \mathcal{M}, \mu)$ is a separable $\sigma$-finite measure space with $\mu$ a complete measure. The separability means that there exists a sequence $\{A_k\}$ of sets in $\mathcal{M}$ such that for $\varepsilon > 0$ and $M \in \mathcal{M}$, there exists a $k$ such that the symmetric difference $A_k \Delta M$ has measure less than $\varepsilon$; equivalently, $L_2(X)$ is a separable metric space. It is then possible to construct on $X$ a particularly simple type of basis called a net structure.

A **net** is a finite or denumerable disjoint collection of measurable sets which cover

$X$. The individual sets in the collection are called **cells.** A sequence $\{\mathcal{N}_k\}$ of nets is called **monotone** provided for every $k$, each cell of $\mathcal{N}_{k+1}$ is a subset of some cell of $\mathcal{N}_k$. The family $\mathcal{N} = \bigcup_{k=1}^{\infty}\mathcal{N}_k$ is called a **net structure.** If $x \in X$, there exists for each $k$ exactly one cell of $\mathcal{N}_k$ containing $x$. We take $N_k \Rightarrow x$ to mean that $x \in N_k \in \mathcal{N}_k$ and $k \to \infty$. Under the above hypotheses on the space, it is always possible to construct such a net structure for which the strong Vitali theorem holds. For development of the theory of differentiation with respect to net structures, see [49], [95], [144], [148], and [167].

Net structures form a special type of cell structure studied by a number of authors, [78], [116], [119], [121], [117], and [92]. Since even a short introduction would involve a number of axioms and definitions, we do not provide any details here and, instead, refer the reader to Rutovic and Pauc [138] for a detailed treatment. We mention only that the basic idea is to generalize the notion of interval function. In [138] one finds generalizations of the theorems of Ward mentioned in Section 5.3 to abstract measure spaces. In spite of the generality of the setting, Misik [92] has shown that certain of the mean value theorems and Darboux properties for derivatives of cell functions hold under quite general conditions.

6.4. **Existence of differentiation bases with Vitali properties.** Since the ability of a differentiation basis to differentiate integrals of a class of functions is closely related to the type of Vitali property the basis possesses, it is natural to ask whether *every* $\sigma$-finite measure space possesses a basis for which the strong Vitali property holds (relative to definition (A) of a Vitali cover). We already mentioned in Section 6.3 that if the space is separable, it possesses a net structure possessing the strong Vitali property. In [74] Kametani and Enomoto constructed basis with the strong Vitali property for an arbitrary locally compact $\sigma$-compact metric space. But what about the general case? Kölzow has proved that each $\sigma$-finite measure space has a differentiation basis of a certain specific type for which the strong Vitali property holds. In fact, one need not even assume that the space be $\sigma$-finite! Kölzow obtained a number (12) of equivalent conditions on a measure space, one of which being the existence of a differentiation basis with the strong Vitali property. (Kölzow's measure spaces have the property that the measure is its own Carathéodory extension and therefore is complete.)

Since Vitali properties are so fundamental to this article, we shall give a proof of the fact that a complete $\sigma$-finite measure space $(X, \mathcal{M}, \mu)$ has a differentiation basis. Suppose we wished to prove this statement. A first attempt might go along the following lines. We observe that the $\sigma$-finiteness assumption allows us, without loss of generality, to assume $\mu(X) < \infty$. Consider now the family $\mathcal{M}^+$ of all sets of positive measure. We would like, if possible, to find a notion of $\Rightarrow$ for which $(\mathcal{M}^+, \Rightarrow)$ has the strong Vitali property. A natural first attempt would be, for each $x \in X$, to order those sets in $\mathcal{M}^+$ containing $x$ by downward inclusion: $I_1$ is beyond

$I_2$ provided $I_1 \subset I_2$. If $\mathscr{M}_x^+$ denotes the family of all such sets, we would like to take $\Rightarrow$ to be contraction by inclusion as defined in Section 6.1. We immediately see a difficulty—we do not have a directed system, because the intersection of two sets of positive measure containing a point might fail to be of positive measure.

We can handle this difficulty as follows. A theorem of von Neumann-Maharam [105], [83] states that every complete measure space of finite measure possesses a **linear lifting**. More precisely, there exists a mapping $L: \mathscr{M} \to \mathscr{M}$ such that

(i) $\mu(M \Delta L(M)) = 0$ for all $M \in \mathscr{M}$,

(ii) $L(M_1) = L(M_2)$ whenever $\mu(M_1 \Delta M_2) = 0$,

(iii) $L(M_1 \cap M_2) = L(M_1) \cap L(M_2)$ for all $M_1, M_2 \in \mathscr{M}$,

(iv) $L(M_1 \cup M_2) = L(M_1) \cup L(M_2)$ for all $M_1, M_2 \in \mathscr{M}$,

(v) $L(\varnothing) = \varnothing$, $L(X) = X$.

Thus, $L$ picks out one member from each equivalence class of measurable sets in such a way as to be "linear" on finite intersections and unions, as well as to preserve the empty set and the whole space.

Now, instead of considering the class $\mathscr{M}^+$, consider the class $\mathscr{I} = \{I \in \mathscr{M}^+ : I = L(I)\}$. It is easy to verify that for every $M \in \mathscr{M}$, $L(L(M)) = L(M)$; thus, $\mathscr{I}$ consists of all "lifted" sets. For $x \in X$, let $\mathscr{I}_x$ consist of those $I \in \mathscr{I}$ which contain $x$. That this family can be directed by inclusion follows immediately from conditions (iii) and (v). Thus, we take $I_\alpha \Rightarrow x$ to mean $x \in I_\alpha \in \mathscr{I}_x$, and the sequence $\{I_\alpha\}$ is cofinal with $\mathscr{I}_x$; in our language, $\Rightarrow$ is "contraction by inclusion".

We now show that $(\mathscr{I}, \Rightarrow)$ possesses the strong Vitali property, a Vitali cover being in accord with definition (A) of Section 6.3.

Let $A \subset X$ with $\mu^*(A) > 0$, where $\mu^*$ denotes the outer measure determined by $\mu$. Let $\mathscr{J}$ be a Vitali cover of $A$. Let $\bar{A}$ be a measurable cover for $A$ and let $B = L(\bar{A})$. Let $x \in A \cap B$. Then $B \in \mathscr{I}_x$. Since $\mathscr{J}$ is a Vitali cover for $A$, there exists $J \in \mathscr{J}$ such that $x \in J \subset B$. We have shown that there exists $J \in \mathscr{J}$ such that $J \subset B$. Consider the family $\mathscr{A}$ of all sets in $\mathscr{J}$ which are contained in $B$. A subfamily $\mathscr{A}_1 \subset \mathscr{A}$ is called **admissible** if each pair of its members is disjoint. Partially order the admissible subfamilies by (upward) inclusion: $\mathscr{A}_1$ is beyond $\mathscr{A}_2$ if $\mathscr{A}_1 \supset \mathscr{A}_2$. Since $\mu(B) < \infty$, each admissible family is at most denumerable. Now each chain of admissible families has an upper bound (its union) which is also an admissible family, and therefore denumerable. By Zorn's Lemma, there exists a maximal admissible family. Denote its members by $I_1, I_2, \cdots$. We show that the sequence $\{I_k\}$ has the desired properties. It is clear that the members of $\{I_k\}$ are pairwise disjoint. We show $\mu(B \sim \bigcup I_k) = 0$. Since $\bigcup I_k$ is a finite or denumerable union of sets in $\mathscr{I}$ and $B$ is measurable, the set $B \sim \bigcup I_k$ is also measurable. Suppose $\mu(B - \bigcup I_k)$ were positive. Let $M = L(B - \bigcup I_k)$. Then $M$ is a set of positive measure disjoint from each $I_k$, because $\mu(M) = \mu(B - \bigcup I_k)$, and the set $M$ as well as each $I_k$, is a lifted set. (If two lifted sets intersect, they intersect in a set of positive measure.) Let $y \in M \cap A$. Then $M \in \mathscr{I}_y$ and $L(M) \subset B$. Thus, the family $M, I_1, I_2, \cdots$ is an admissible family, contradicting the maximality of the family $I_1, I_2, \cdots$.

Observe that $\bigcup I_k \subset B$ and $B$ is equivalent to $\tilde{A}$, a measurable cover for $A$. Thus, the sequence $I_1, I_2, \cdots$ has a *zero measure overflow*. We have proved the following theorem.

EXISTENCE THEOREM. *If $(X, \mathscr{M}, \mu)$ is any complete totally $\sigma$-finite measure space, there exists a differentiation basis for which the strong Vitali theorem holds. The $\varepsilon$-overflow requirement can be replaced with a zero overflow requirement.*

Kölzow [76] actually showed that a measure space which is its own Carathéodory extension and meets a certain non-triviality condition, has a strong Vitali basis if and only if the space admits of a linear lifting. One need not assume $\sigma$-finiteness. A number of other conditions are equivalent to these two. One such condition is that the space be **decomposable**; that is, there exists a disjoint family of sets of positive measure with the property that each set of positive measure intersects one of these sets in a set of positive measure. Another equivalent condition is that a specific version of the Radon-Nikodym theorem holds.

Thus we have answered the question posed in the introduction: In what sense and under what circumstances is the Radon-Nikodym derivative a pointwise derivative? The answer, briefly, is that every complete measure space for which the Radon-Nikodym theorem holds, admits a differentiation basis which differentiates the integrals of every summable function to its Radon-Nikodym derivative.

6.5. **Miscellaneous remarks.** We end Chapter 6 with a few brief remarks.

Several authors have considered the notion of approximate continuity of functions in the abstract setting, [136], [84], [54], [151]. The definitions parallel those in the classical setting, and we shall not present a development here. We mention only that an analogue of the theorem stated in Section 5.2 holds in metric measure spaces with the differentiation basis of "closed spheres centered at the point" [136]. It is not difficult to verify that the theorem applies in more general settings as well.

We already mentioned in Section 6.3 that Darboux properties for derivatives of set functions have been established in abstract spaces.

In addition to the works cited already in Chapter 6, a number of other results dealing with differentiation in abstract spaces can be found in [2], [3], [11], [23], [43], [44], [45], [50], [55], [56], [57], [68], [77], [79], [80], [107], [110], [111], [112], [113], [114], [122], [123], [124], [134], [153], [166].

# VII. APPLICATIONS

The theory of differentiation of integrals and measures has certain applications to other parts of mathematics. We already saw, in Chapter 4, application of the theory in Euclidean spaces, and in Section 5.5 we gave a brief introduction to an infinite dimensional space that has uses in probability theory. We now consider further applications of the theory in the abstract setting.

7.1. **Continuity in measure spaces** [18]. An abstract measure space does not necessarily come furnished with a topology, so one does not in general have a notion of continuous function. Nevertheless, one can use a differentiation basis to obtain a reasonable notion of continuous function. Consider for a moment the one dimensional Lebesgue space furnished with the usual bases $(\mathscr{I}, \Rightarrow)$. Let $f$ be summable on sets of $\mathscr{I}$, and let $\sigma(E) = \int_E f\,d\mu$ for each $E$ of finite measure. Then $\lim_{I \Rightarrow x} \sigma(I)/\mu(I) = f(x)$ a.e. Even if the equality holds everywhere this does not imply that $f$ is continuous. It only implies that $f$ is everywhere a derivative. Suppose, however, that we write $E \Rightarrow x$ if $x \in E$, $\mu(E) > 0$, and $\delta(E) \to 0$. It is then easy to verify that $f$ is continuous if and only if $\lim_{E \Rightarrow x} \sigma(E)/\mu(E) = f(x)$ for every $x$. (Compare with the relationship between approximate continuity and regular derivative. In the present case, no regularity is assumed.) In considering other standard examples of topologized measure spaces, one observes similar results. These observations motivate us to a general notion of continuity.

Let $(X, \mathscr{M}, \mu)$ be a measure space furnished with a differentiation basis $(\mathscr{I}, \Rightarrow)$. Let $\mathscr{C}$ be the class of functions $f$, summable on sets of $\mathscr{I}$, for which $\lim_{E_\alpha \Rightarrow x} \sigma(E_\alpha)/\mu(E_\alpha) = f(x)$ for all $x \in X$. Here $E_\alpha \Rightarrow x$ means for each $\alpha$ there exists $I_\alpha \in \mathscr{I}$ such that $E_\alpha \subset I_\alpha$ and $I_\alpha \Rightarrow x$.

Now, let $\tau$ be the smallest topology on $X$ for which every function in $\mathscr{C}$ is continuous. One can prove without difficulty that $\mathscr{C}$ is exactly the class of $\tau$-continuous functions.

Because of the extreme generality of $\Rightarrow$, a great deal of pathology can exist, even if $(\mathscr{I}, \Rightarrow)$ possesses the strong Vitali property. For example, $\tau$ might be trivial. Under certain circumstances, however, the class of continuous functions is large. For example, if $(X, \mathscr{M}, \mu)$ is a separable measure space, one can construct a net structure on $X$ such that the resulting topology generated by the family $\mathscr{C}$ is pseudometrizable and **compatible** with $\mu$. Here the term "compatible" means that each Borel set is measurable, the measure $\mu$ is "almost" regular, and the class $\mathscr{C}$ is sufficiently large for the following form of Lusin's theorem to hold.

THEOREM. *Under the conditions stated above, if $f$ is measurable and $\varepsilon > 0$, there exists a measurable set $A$ and a continuous functon $g$, such that $\mu(X - A) < \varepsilon$ and $g = f$ on $A$.*

One cannot, however, require the set $A$ to be closed in the statement of the theorem. We mention that the de la Vallée Poussin theorem [144: p. 155] is valid:

THEOREM. *If $\sigma$ is a finite countably additive set function defined at least on the Borel sets, if $D\sigma$ denotes its derivative with respect to the net structure mentioned above, and if $E_\infty = \{x : D\sigma(x) = \infty\}$, $E_{-\infty} = \{x : D\sigma(x) = -\infty\}$, then for every Borel set $E$,*

$$\sigma(E) = \sigma(E \cap E_\infty) + \sigma(E \cap E_{-\infty}) + \int_E D\sigma(x)\, d\mu .$$

In particular, if $\sigma$ is singular so that $D\sigma = 0$ a.e., and if $X$ is non-atomic with respect to $\sigma$, one sees immediately from the de la Vallée Poussin theorem that $D\sigma$ must be infinite on a non-denumerable set.

The de la Vallée Poussin theorem does not hold with respect to arbitrary differentiation bases. For example, if for every Borel set $E \in R_2$, $\sigma(E)$ is the one dimensional Lebesgue measure of $E \cap X$, where $X$ is the "x-axis" and $(\mathscr{I}, \Rightarrow)$ is the basis consisting of closed disks, where $I \Rightarrow x$ means $x \in I$ and $\delta(I) \to 0$, then $D\sigma(x)$ is never infinite. In fact, $D\sigma \equiv 0$ off $X$, $\bar{D}\sigma = +\infty$, and $\underline{D}\sigma = 0$ on $X$. Thus if the theorem held, we would have $\sigma \equiv 0$, a contradiction.

We saw in Section 6.4 that every complete $\sigma$-finite measure space has a differentiation basis possessing the strong Vitali property. If the space is already topologized by a topology $\mathscr{T}^*$, it may or may not be the case that the topology $\mathscr{T}$ obtained from the basis is the same as $\mathscr{T}^*$. We pose the problem: Under what circumstances does a topological measure space with topolgy $\mathscr{T}^*$ possess a differentiation basis that gives rise to $\mathscr{T}^*$ and which possesses the strong Vitali property?

7.2. **Functional differentiation systems.** Let us begin this section by casting the classical Lebesgue theorem in a somewhat different form which is suggestive of a certain type of generalization. Using the notation of Section 2.1, we write

$$\frac{\sigma(I)}{\mu(I)} = \frac{\int_I f\, d\mu}{\mu(I)} = \frac{\int f\phi_I d\mu}{\int \phi_I d\mu} = \frac{\int \phi_I\, d\sigma}{\int \phi_I\, d\mu},$$

where $\phi_I$ denotes the characteristic function of $I$, and the symbol $\int$ denotes integration over the whole space. Thus, the Fundamental theorem takes the form

$$\lim_{I \Rightarrow x} \frac{\sigma(I)}{\mu(I)} = \lim_{F_x} \frac{\int \phi_I\, d\sigma}{\int \phi_I\, d\mu} = f(x) \text{ a.e.}$$

where $\sigma$ is the indefinite integral of $f$, $\mathscr{F}_x$ is the family of characteristic functions of intervals containing $x$, and $\lim_{\mathscr{F}_x}$ means that the limit is taken as $I \Rightarrow x$.

The reason for writing our theorem in such a manner is that the result holds in many cases when the families $\mathscr{F}_x$ are different from the ones appearing above. For example, if for each $x \in [0, 2\pi)$ we let $P_r(x, \cdot)$ denote the Poisson kernels centered at $x$, we know from Fatou's theorem (Section 4.3) that

$$\lim_{r \to 1} \frac{\int P_r(x,t) f(t) dt}{\int P_r(x,t) dt} = \lim_{r \to 1} \frac{\int P_r(x,t) d\sigma(t)}{\int P_r(x,t) d\mu(t)} = f(x) \text{ a.e.}$$

Similar results are valid for many other kernels which are approximate identities, and each of these results can be cast in a form involving a limit of a quotient of integrals of "kernel" functions with respect to two different measures that are a.e. the Radon-Nikodym derivative of the measure appearing in the numerator with respect to the measure appearing in the denominator.

In [127] and [128] de Possel gave conditions, in a very abstract setting, that a deriving filter of kernels sums all functions in certain classes, that is, reproduces the Radon-Nikodym derivative as a limit of a quotient as described above. We shall state one of de Possel's results and indicate how it can be used to obtain theorems concerning summability methods.

Let $(X, \mathscr{M}, \mu)$ be a sigma-finite measure space which (for simplicity) we assume to be complete. Let $\mathscr{M}'$ denote the class of sets of finite measure. For any set $A$, we shall denote by $\bar{A}$ a measurable cover for $A$ and by $\phi_A$ the characteristic function of $A$. Let $U$ denote the family of finite non-negative measurable functions $f$ which vanish outside some set of $\mathscr{M}'$ and for which $\int f d\mu > 0$. Let $S$ denote a system of filters obtained by associating to each point $x \in X$ a filter $\mathscr{F}_x$ on $U$, and let $\sigma$ denote an absolutely continuous vector valued measure taking values in a Banach space. If $\sigma(E) = \int_E g\, d\mu$ for every $E \in \mathscr{M}'$, we ask for conditions under which

$$g(x) = \lim_{\mathscr{F}_x} \frac{\int f\, d\sigma}{\int f\, d\mu} \text{ a.e.}$$

When this equality holds we say $S$ derives $\sigma$. If $S$ derives every $\sigma$ for which $\| \sigma(E) \| / \mu(E)$ is bounded, we say $S$ is a **weak differentiation system.** If $S$ derives every absolutely continuous $\sigma$, we say $S$ is a **strong differentiation system.** In [127], de Possel gave conditions that a system $S$ be a strong or weak differentiation system. His conditions are a bit complicated, but some insights to the conditions can be obtained by considering the case where the functions in the filters $\mathscr{F}_x$ are characteristic functions of sets in a differentiation basis, with convergence of the filters induced from contraction of the sets. In that case, the conditions reduce to density or Vitali type conditions.

THEOREM 1. *The following conditions are equivalent:*

(A) *$S$ is a weak differentiation system.*

(B) *For each system $V$ obtained by associating with each point $x \in A$ a set $F_x$ of functions meeting all the sets of $\mathscr{F}_x$, where $A \subseteq X$ is any set satisfying $0 < \mu(\bar{A}) < \infty$, and for $\varepsilon > 0$, there exists a point $x \in A$, a function $\psi$ in $F_x$, and a number $\lambda > 0$ such that*

$$\int \min(\phi_{\bar{A}}, \lambda\psi)d\mu \geqq (1-\varepsilon)\int \lambda\psi d\mu.$$

(C) *For each $V$, $A$ and $\varepsilon$ as in* (B), *there exist finite sets of points $x_i \in A$, functions $\psi_i \in F_{x_i}$, and numbers $\lambda_i > 0$ such that*

$$\int \left| \phi_{\bar{A}} - \sum \lambda_i \psi_i \right| d\mu < \varepsilon.$$

De Possel also stated a somewhat more complicated condition for a system to be a strong differentiation system, and, in [128], he d veloped an even stronger notion of **complete differentiation system**, in which the measure $\sigma$ does not have to be absolutely continuous and for which the set function $\nu$ given by $\nu(E) = \sigma(E) - \int_E g\, d\mu$ is singular. We do not give any details here.

Theorems of the de Possel type take much simpler forms if one specializes the conditions somewhat.

THEOREM 2 [19]. *Let $(\mathscr{I}, \Rightarrow)$ be a differentiation basis for the finite measure space $(X, \mathscr{M}, \mu)$. Suppose with each point $x_0 \in X$ there is associated a generalized sequence $\{\psi_\alpha(x_0;x): \alpha \in A_{x_0}\}$ of functions in $L_1(X)$. Suppose further that*

(i) $\psi_\alpha(x_0;x) \geqq 0$ *for* $x_0 \in X$ *and* $\alpha \in A_{x_0}$,

(ii) $\lim_\alpha \int_X \psi_\alpha(x_0;x)d\mu = 1$ *for all* $x_0 \in X$.

*Then a necessary and sufficient condition that for every $f \in L_\infty(X)$*

$$\lim_\alpha \int_X \psi_\alpha(x_0;x)f(x)d\mu = f(x_0),$$

*for almost every $x_0 \in X$ (respectively, for each point $x_0$ of approximate continuity of $f$) is*

$$\text{(iii)} \qquad \lim_\alpha \int_{X\setminus M} \psi_\alpha(x_0;x)d\mu = 0,$$

*for almost every point of each measurable set M (respectively,*

(iii′) $$\lim_\alpha \int_{X\setminus M} \psi_\alpha(x_0; x)d\mu = 0$$

*for every measurable set M having $x_0$ as a point of density).*

We note that conditions (i) and (ii) correspond to conditions often required of approximate identities in classical settings, while conditions (iii) or (iii′) replace the standard conditions that eventually the kernels "become small" outside any interval containing $x_0$, with the condition that eventually the kernels become small outside any set having $x_0$ as a point of density.

We indicated near the end of Section 4.2 that there is a striking similarity between strong and ordinary differentiation on the one hand, and unrestricted and restricted $(C, 1)$ summing of multiple Fourier series on the other. Similar results hold in other areas of analysis, involving the possibility of taking limits in an unrestricted or restricted manner.

One can look at these similarities in terms of functional differentiation systems. Consider, for example, the case of the square $[0, 2\pi] \times [0, 2\pi]$. The Fejer kernels and the differentiation kernels $\phi_I/\mu(I)$, where $\phi_I$ is the characteristic function of the interval $I$, possess similar properties as functional differentiation systems. If one considers all characteristic functions of two dimensional intervals, or all Fejer kernels, the resulting systems are weak (but not strong!) differentiation systems. On the other hand, if one considers only the characteristic functions of intervals whose side lengths $s$ and $t$ satisfy, say, $1/N = s/t \leqq N$, for some positive integer $N$, or if one considers only those Fejer kernels $K_{nm}$ for which $1/N \leqq n/m \leqq N$, then the resulting systems are strong differentiation systems. The similarity that one sees between certain results one obtains in the differentiation of integrals and certain results obtained where approximate identities are involved (e.g., the roles the class $L \log L$ plays in the theorem of Jessen-Marcinkiewicz and Zygmund and in multiple Fourier series) might be less surprising when viewed in terms of the de Possel theory outlined above.

Results relating summing by approximate identities to differentiation theory in terms of the "shapes" of the kernels can be found in [19]. They indicate that if the kernels have certain shapes relative to $(\mathscr{I}, \Rightarrow)$, then they sum whenever the differentiation kernels sum. The results are analogues of the classical theorems of Romanovsky and Faddeyev (see Natanson [101]). The theorems are a bit too complicated to state fully here. Roughly speaking, they state that under rather general conditions on $(X, \mathscr{M}, \mu)$ and $(\mathscr{I}, \Rightarrow)$ and rather standard conditions on the kernels, if each kernel $\psi(x_0; x)$ associated with $x_0 \in X$ is **increasing**, then the kernels sum each $f \in L_1(x)$ at every point $x_0$ for which $\lim_{I \Rightarrow x_0} 1/\mu(I) \int_I |f(x_0) - f(x)| d\mu = 0$ (i.e., at every Lebesgue point). The term "increasing" means that if $I$ and $J$ are in a generalized sequence contracting to $x_0$ with $I$ beyond $J$ and $J \supset I$, then

$$\sup\{\psi(x_0; x): x \in J \backslash I\} \leqq \inf\{\psi(x_0; x): x \in I\}.$$

(This is an analogue of Romanovsky's theorem. An analogue of Faddeyev's theorem is also valid. Here one requires only that the kernels have increasing majorants, where the integrals of the majorants associated with each $x_0$ are uniformly bounded.) As a corollary, one has that such increasing kernels will sum all $L_1$ functions if $(\mathscr{I}, \Rightarrow)$ has the strong Vitali property and will sum all $L_\infty$ functions if $(\mathscr{I}, \Rightarrow)$ has the weak Vitali property. Thus the class of functions such increasing kernels will sum is closely related to the class of functions whose integrals $(\mathscr{I}, \Rightarrow)$ differentiates. Actually, one obtains the conclusion on a set of points which satisfy a property that is less restrictive than the property of being a Lebesgue point.

One can also prove a product theorem [19] which states that, under rather general circumstances, if a family of kernels sums all $f \in L_\infty(X, \mathscr{M}, \mu)$ a.e. and another family of kernels sums all $f \in L_\infty(Y, \mathscr{N}, \nu)$ a.e., then the product kernels sum all $L_\infty$ functions in the product of the two spaces a.e. In particular, the result applies to $R$ with the usual interval basis, Lebesgue measure, and the Fejer kernels, thereby verifying that the double Fourier series of an arbitrary $f$ essentially bounded on the square $[0, 2\pi] \times [0, 2\pi]$ converges unrestrictedly $(C, 1)$ to $f$ a.e. Similar results hold for the Poisson kernel, thereby obtaining a weak form of a theorem of Tsugi [159; p. 140]. Applying the product theorem to the differentiation kernels, one obtains a weak form of the product density theorem mentioned near the end of Section 6.2.

Further indications of the role differentiation theory plays in the theory of approximate identities can be found in a number of places. We mention in particular Edwards and Hewitt [36]. In that article the setting is that of a locally compact group. The differentiation basis consists of a sequence of sets, measurable with respect to a left Haar measure, and its translates. If this sequence satisfies certain conditions, then a theory (involving such things as a covering lemma, a Hardy-Littlewood maximal theorem, and the like) can be developed. In particular, sequences satisfying these conditions exist in every Lie group and in every finite dimensional compact group. The differentiation theory is then used to obtain a number of theorems concerning the pointwise limits of sequences of convolution operators.

In closing this section, we mention that the author is indebted to Professor C. Y. Pauc for calling to his attention the role of *functional differentiation systems.* See also Pauc's remarks in [120: p. 147–8].

7.3. **Other applications.** We mention briefly certain other areas of applications o the abstract theory.

Using differentiation bases, Trjitzinsky [155], [157], [158] has extended the notion of Denjoy-Perron integration to abstract measure spaces. The referenced works are very technical in nature, and the type of differentiation theory used is related to the Denjoy approach [31] (cf. Section 6.2D). We shall not develop any details of the theory here. We mention only that many of the classical results on the Denjoy and Perron integrals have analogues in the abstract setting.

For relationships between differentiation theory and the theory of martingales, see Krickeberg and Pauc [80], and Hayes and Pauc [63].

In addition, a number of the tools of differentiation theory (covering theorems, Hardy-Littlewood maximal function, etc.) have uses in many situations involving abstract spaces as well as in situations involving the Euclidean spaces.

## VIII. PROBLEMS

We close with a discussion of several problems which appear to be unsolved and which can easily be stated in terms of the material presented in Chapters 1–7 above. The numbers in parentheses indicate the sections in which relevant discussion can be found.

(2.2) To the best of the author's knowledge, Table 1 on page 10 includes all known relations among Vitali properties, halo properties, and the class of functions whose integrals a given basis differentiates. There are, therefore, a number of questions remaining to be answered. For example, is there a Vitali or halo property which is necessary and sufficient for a basis to differentiate the integral of every $L_p$ function, with $1 < p < \infty$? What kinds of halo properties are relevant if one drops or weakens the requirements on the differentiation bases which appear in the chart (e.g., openness or closedness of the sets in $\mathscr{I}$, the closedness of $\mathscr{I}$ under homothetic transformations etc.)?

(3.2) What is a necessary and sufficient condition for a differentiation basis to be perfect? Similarly, what are necessary and/or sufficient conditions for a basis to have the density property with respect to every Lebesgue-Stieltjes measure?

(5.1) In Section 5.1 we indicated how several properties possessed by derivatives of real functions generalize when we deal with derivatives of measures. There are, of course, many more questions of this type. We shall list three such questions.

a) S. Marcus [86] posed the problem of determining what type of Darboux properties Jacobians and hyperbolic derivatives possess. Since the hyperbolic derivative is (in our language) just the strong derivative, we know that hyperbolic derivatives possess the kind of Darboux properties contained in Misik's result. Similarly, the Jacobian, $J_T$, of a transformation $T: R_n \to R_n$ possesses certain Darboux properties if $J_T$ can be looked upon as a derivative with respect to the family of cubes. This will happen if $T$ is suitably behaved, but not in general. Whether the Jacobian of every transformation (where the Jacobian is defined at each point) has such a Darboux property, seems to be an open question.

There are also a number of other questions we can ask concerning Darboux type properties. For example, Morse [99] has found a certain property of this type for Dini derivates of continuous functions. What are the analogues for the extreme derivates of continuous measures?

b) The problem of characterizing those functions which are derivatives (or upper or lower derivates) of some measure, in terms of metric or topological properties, is open. See [17] for a discussion of the problem for derivates of real functions. The problem of characterizing those functions which are *almost everywhere* derivatives of some continuous measure has been solved by Saks [142] for derivatives with respect to the intervals basis in $R_n$, but not for differentiation bases in general (cf. Section 4.1).

c) Analogues of the theorems of Young, Neugebauer and Denjoy-Young-Saks discussed in Section 5.1 have been found only under rather restricted conditions on $(\mathscr{I}, \Rightarrow)$. It should be possible to obtain similar results in much more general settings.

(5.2) In giving conditions under which the bounded approximately continuous functions can be characterized as the bounded regular derivatives, Rosenthal dealt with metric spaces and with a specific meaning for contraction. To what extent do his results hold in more general settings?

(7.1) In addition to the problem posed at the end of Section 7.1, one can pose a number of problems. A list of such problems appears in [18: p. 256].

(7.2) Since the theory of differentiation of integrals can be looked upon as a special case of the theory of approximate identities, many differentiation theorems have possible analogues in the more general theory. We list a few such problems.

a) As de Possel [127], [128] showed, certain Vitali type theorems have analogues in the setting of approximate identities. De Possel's results were very general. There appear to be many questions here that are still open. For example, are there analogues of all the Vitali and halo type theorems which appear in Section 2.2? It seems likely that there should be a number of such theorems, particularly if one places certain restrictions on the kernels which "tie them down" to a differentiation basis.

b) Can the product theorem mentioned in Section 7.2 be improved? Specifically, are there conditions under which the product kernels sum at every Lebesgue point?

c) The Fourier series of the function $f$ defined by

$$f(x) = \begin{cases} 0 \\ 1 \\ 2 \end{cases} \text{for} \begin{cases} 0 \leqq x < x_0 \\ x = x_0 \\ x_0 < x < 2\pi \end{cases}$$

converges to $f$ everywhere. The same is true of the symmetric derivative of the integral of $f$. Now $f$ is not a Darboux function. On the other hand, if the ordinary derivative of a function exists everywhere, this derivative must have the Darboux property. In terms of approximate identities, there is therefore a difference between the symmetric differentiation, Dirichlet, or Fejer kernels on the one hand and the differentiation kernels on the other. One immediately apparent difference is that the symmetric derivative kernels are biased towards the center of an interval, whereas the differentiation kernels are not. In other words, the differentiation kernels associated with a point $x_0$ include many of the symmetric differentiation kernels which are associated with nearby points, and it is this property which makes it much harder for a derivative to exist than for a symmetric derivative.

A similar situation exists when one is dealing with analytic or harmonic functions defined on an open disk in $R_2$, and is concerned with radial limits on one hand and Stolz angle (or unrestricted) limits on the other. In this latter case, one point $\theta_0$ "borrows" some of the Poisson kernels associated with nearby points. There are a number of questions that can be asked. For example, under what conditions on a family of kernels will the limit functions, assumed to exist everywhere, have a Darboux property? An answer might involve some sort of overlap of the kernels associated with nearby points.

d) The converse of Faddeyev's theorem holds in $R_1$. Under what circumstances does it hold in the general setting?

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University of California,
Santa Barbara, California