Preface

heu-ris-tic [adjective]

1. serving to indicate or point out; stimulating interest as a means of furthering investigation.
2. encouraging a person to learn, discover, understand, or solve problems on his or her own, as by experimenting, evaluating possible answers or solutions, or by trial and error: a heuristic teaching method.
[Source: Dictionary.com]

Introduction

This book is an outgrowth of classes given at the University of California,Santa Barbara, mainly for students who had little mathematical background. Many of the students indicated they never understood what mathematics was all about (beyond what they learned in algebra and geometry). Was there any more mathematics to be discovered or created? How could one actually discover or create new mathematics?

In order to give these students some sort of answers to such questions, we designed a course in which the students could actually participate in the discovery of mathematics. The class was not presented in the usual lecture fashion. And it did not deal with topics that the students had seen before. Ordinary algebra, geometry, and arithmetic played minor roles in most of the problems we addressed. Whatever algebra and geometry that did appear was relatively easy and straightforward.

Our objective was to give the students an appreciation of mathematics, rather than to provide tools they would need in some field that required mathematics. In that sense, the course was like a course in music appreciation or art appreciation. Such courses don't attempt to train students to become pianists, composers, or artists. Instead, they attempt to give the students a sense of the subject.

Why do so many intelligent people have so little sense of the field of mathematics? A partial explanation involves the difficulty in communicating mathematics to the general public. Without special training in astronomy, medicine, or other scientific areas, a person can still get a sense of what goes on in those areas just by reading newspapers. But this is much more difficult in mathematics. This may be so because much of modern mathematics involves very technical language that is difficult to express in ordinary English. Even professional mathematicians often have difficulty communicating their work to other professional mathematicians who work in different areas.

This isn't surprising when one realizes how many areas and sub areas there are in mathematics. *Mathematical Reviews* (MR) is a journal that provides short reviews of mathematical papers that appear in over 2000 journals from around the world. The subject classification used by MR has over 50 subject areas, each of which has several subareas. Each of these subareas has many sub-sub areas. A research mathematician might be an expert in several of the sub-sub areas, be conversant in several areas, and know very little about the other areas.

Objectives

Our objective is to impart some of the flavor of mathematics. We do this in several ways. First, by actively participating in the discovery process, a reader will get a sense of how mathematicians discover new mathematics.

A problem arises. Discovery often begins with some experimentation to help give a sense of what is involved in the problem. After a while one might have enough understanding of the problem to be able to make a plausible conjecture, which one then tries to prove. The attempt to prove the conjecture can have several different outcomes. Sometimes the proof works. Other times it doesn't work, but in trying to prove it one learns much more about the problem and identifies some stumbling blocks.

Sometimes these stumbling blocks seem insurmountable and one tries to prove they actually *are* insurmountable—the conjecture is false. That may create its own stumbling blocks. All the time one learns more and more about the problem. Finally one either proves the conjecture or disproves it. (Or simply gives up!).

We shall see all of this unfolding in the several chapters in the book. Our discovery process will be similar to that of a research mathematician's, though our problems will be much less technical.

The first part of each chapter deals with a problem we wish to consider. We then go into the discovery mode and eventually obtain some answers. After this we turn to other aspects of mathematics related to the material of the chapter. What is the history of the problem? Who solved it? What are some related problems ? How can other areas of mathematics be brought to bear on the problem? Do computers have any role in solving the problems raised? What about conjectures that seemed to be true, but were eventually proven false? Or remain unsolved?

We have tried to find some balance between discovery and instruction. This is not always possible: it is impossible to resist the many occasions when some idea leads naturally to another wonderful idea. The reader will not discover the connection, even with prodding, so we drop our heuristic approach and explain the new ideas. This is probably in the nature of things. When we look back on everything we have learned, certainly it is all a combination of stuff we figured out for ourselves and other stuff that we learned from others. It is the combination of the two that makes learning rewarding and productive. It is likely the stress on just the instruction part that explains the many people in this world who claim to dislike or fear mathematics.

Prerequisites

The main prerequisite for getting much from this book is curiosity and a willingness to attempt the problems we present. These problems usually set things up for the next stage in the discovery process. This is different from most text books, where the problems at the end of a section are intended to firm the readers' knowledge of the material just presented.

Almost all problems have answers supplied at the end of the chapter. The word ANSWER following a problem indicates that an answer is supplied. For readers using a PDF file on a computer or laptop screen, that word is hyperlinked to the answer. Readers working on a paperback version will have to scan the end of the chapter to find the appropriate answer.

When the book is read in a self-study manner, rather than in a classroom setting with an instructor to set the pace, there may be a temptation to move ahead quickly, to get to the end of the process to learn the result. (Did the butler commit the crime?). We urge that one resist the temptation. The students who got the most out of the class were the ones who participated actively in the discovery process. This included working the problems as they arose. They said that understanding this process was of more value to them than learning the answer.

In order to understand the material in most of the chapters, one needs a bit of algebra (just enough to be able to manipulate some simple algebraic expressions, though such manipulations play only a very minor role), a bit of geometry, and a little arithmetic.

One topic that is not usually covered in a first course in algebra is *mathematical induction*. This tool appears in several places. Readers not familiar with mathematical induction can reasonably work through a chapter that has an induction argument until that argument is needed. At that point, one can consult the Appendix where induction is discussed and induction proofs are given that are relevant to various problems we discuss. Induction does not take part in the discovery process—it is used only to verify that certain statements are true.

Rigor versus intuition

Professional mathematicians must be rigorous in their work. This involves giving careful definitions, even of apparently familiar objects. This often involves a great deal of "technical machinery." A mathematician needs to know such things as *exactly* what a "curve" is, what it means to "go around a curve so that the inside is to the left," how to mathematically describe the number of "holes" in a pretzel and the meaning of area.

It should be understood, however, that this is not the situation when a mathematician first starts thinking of a problem and working out a solution. Things are rather vague and intuitive in the early stages. The polish and rigor appear in full force only in the final drafts.

Since this book is not intended for mathematicians, who would require formal definitions and proofs, we can relax these requirements considerably. Everything we say in an informal way *can* be said in a mathematically rigorous way, but that is not our purpose. Our purpose is to provide some of the flavor of mathematics and introduce the reader to topics that some students were surprised to find involved mathematics. Thus we can take for granted that readers intuitively understand concepts such as curves, inside, left, holes, and area. We will occasionally describe a concept with which the reader may not be familiar, but our overall style is primarily a leisurely, informal one.

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Most of the figures were prepared by us using *Mathematica*.™ A number of the figures were found on the internet and, naturally enough, it has proved to be difficult to give proper attribution. Any person who is the original author of such figures is invited to write to us with instructions as to whom to give proper credit. We are thankful for those who have released such figures to the public domain and will be equally thankful to those who wish credit.

We are always grateful for comments and will attempt to incorporate them into future printings (or future editions) with explicit acknowledgement of the sources. Please write to the authors at thomson@sfu.ca.

To the Instructor

One might notice that, on occasion, one or more problems follow after only a short discussion. This occurs when we believe this short discussion already presents an opportunity for the reader to get a sense of how we might continue. When we taught the class, we often found it convenient to make a small amount of progress on each of two chapters in one class session. How this worked in practice varied with what happened in class discussion. Sometimes the material we list as problems actually became part of the class discussion, rather than as problems to be discussed at the next class session. It worked best to be flexible and see where the discussion took us in determining whether we should solve some of the problems in lecture form, or leave them as problems to be discussed in the next class meeting.

In a typical one-quarter term we would have covered four chapters in a leisurely fashion, at least through the discovery of the solution to the main problems of the chapter. We also were able to cover some of the material at the end of the chapters. Available time, class interests, and level of difficulty relative to the students' backgrounds determined what we covered.

We provide answers to most of the problems, in particular to those that point the way to further progress. We leave a few unanswered. Some of these we used as quizzes or homework to be collected.