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ON THE HENSTOCK STRONG VARIATIONAL INTEGRAL

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The theory of integration in division spaces introduced by Henstock ([3], [4]) serves to unite and simplify much of the classical material on nonabsolute integration as well as to provide a new approach to Lebesgue integration. In this paper we sketch a simplified approach to the division space theory and show how it can lead rapidly to the standard Lebesgue-type theory without a substantial departure from the usual methods; some applications to integration in locally compact spaces are briefly developed. No attempt has been made to state the best possible or most general results obtainable: our attention is fixed throughout on the strong variational integral for functions with values in a normed linear space.

1. **Elementary theory of division spaces.** The theory presented here is a special case of that in [4] but which should be adequate for most purposes. Let T be a set and $\mathfrak{I}$ a collection of pairs (I, x) $(I \subseteq T, x \in T)$. A finite subset $\mathfrak{D}$ of $\mathfrak{I}$ is said to be a *division* if the sets in $\{I: (I, x) \in \mathfrak{D}\}$ are disjoint. For a division $\mathfrak{D}$ we write $\sigma(\mathfrak{D}) = \bigcup\{I: (I, x) \in \mathfrak{D}\}$ and we call any set $E = \sigma(\mathfrak{D})$ an *elementary set* and $\mathfrak{D}$ a *division of E*.

If $X \subseteq T$ and $\mathbf{S} \subseteq \mathfrak{I}$ the following subsets of $\mathbf{S}$ are defined:

(1) $$\mathbf{S}(X) = \{(I, x) \in \mathbf{S}: I \subseteq X\}$$

(2) $$\mathbf{S}[X] = \{(I, x) \in \mathbf{S}: x \in X\}.$$

Definition 1. The ordered triple $(T, \mathfrak{A}, \mathfrak{I})$ is said to be a *division space* provided

(i) $\mathfrak{A}$ is a collection of subsets of $\mathfrak{I}$ such that every $\mathbf{S} \in \mathfrak{A}$ contains a division of each elementary set;

(ii) $\mathfrak{A}$ is directed by set inclusion (i.e. if $\mathbf{S}_1$ and $\mathbf{S}_2$ belong to $\mathfrak{A}$ there is an $\mathbf{S} \in \mathfrak{A}$ with $\mathbf{S} \subseteq \mathbf{S}_1 \cap \mathbf{S}_2$).

The division space $(T, \mathfrak{A}, \mathfrak{I})$ is said to be *additive* if

(iii) for every $\mathbf{S} \in \mathfrak{A}$ and each elementary set E there is an $\mathbf{S}^* \in \mathfrak{A}$ such that $\mathbf{S}^* \subseteq \mathbf{S}(E) \cup \mathbf{S}(\backslash E)$. (Note that this implies that the collection of elementary sets forms a ring.)

The division space $(T, \mathfrak{A}, \mathfrak{I})$ is said to be *decomposable* if

(iv) for every sequence $\{X_k\}$ of disjoint subsets of T and each $\{\mathbf{S}_k\} \subseteq \mathfrak{A}$ there is an $\mathbf{S} \in \mathfrak{A}$ such that $\mathbf{S}[X_k] \subseteq \mathbf{S}_k[X_k]$ for each index k.

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The division space $(T, \mathfrak{A}, \mathfrak{I})$ is said to be *fully decomposable* if

(v) for every family $\{\mathbf{S}_x \in \mathfrak{A}: x \in T\}$ there is an $\mathbf{S} \in \mathfrak{A}$ such that $\mathbf{S}[\{x\}] \subseteq \mathbf{S}_x[\{x\}]$, $(x \in T)$.

For the reader unfamiliar with ([2], [3], [4]) and the term "division spaces" (which the referee has suggested would be better rendered as "partition spaces") we indicate the special example in §3 below and the following: Let $T = \mathbf{R}^n$ (n-dimensional Euclidean space) and let $\mathfrak{I}$ be the collection of all pairs $([a, b), x)$ where $x \in \mathbf{R}^n$ and $[a, b)$ is a half-closed rectangle in $\mathbf{R}^n$. For every positive number δ we define $\mathbf{S}_\delta \subseteq \mathfrak{I}$ by $([a, b), x) \in \mathbf{S}_\delta$ if and only if $x \in [a, b]$ and $[a, b)$ is contained in the closed sphere centered at x with radius δ. Then, if $\mathfrak{A} = \{S_\delta : \delta > 0\}$, $(\mathbf{R}^n, \mathfrak{A}, \mathfrak{I})$ is the division space normally associated with the Riemann integral.

If we modify the above by permitting δ to depend on x (i.e. δ is a positive function on $\mathbf{R}^n$) then the resulting division space is fully decomposable and leads to a theory of integration which is more far reaching than even Lebesgue theory in Euclidean space. It is precisely this seemingly trivial modification which underlies the whole theory and which has returned attention to the classical idea of Riemann partitions of sets as a basis for integration.

Let h be a function defined on $\mathfrak{I}$ and with values in a normed linear space $\mathbf{E}$. We define the *variation* of h with respect to a collection $\mathbf{S}(\mathbf{S} \subseteq \mathfrak{I})$:

$$V(h, \mathbf{S}) = \sup (\mathfrak{D}) \sum \|h(I, x)\| \tag{3}$$

where the supremum is taken over all divisions $\mathfrak{D}(\mathfrak{D} \subseteq \mathbf{S})$ and $(\mathfrak{D}) \sum$ denotes a summation over all $(I, x) \in \mathfrak{D}$, an empty sum by convention being zero.

If $\mathfrak{A}$ is a family of subsets of $\mathfrak{I}$ then the variation of h with respect to $\mathfrak{A}$ is also defined:

$$V(h, \mathfrak{A}) = \inf \{V(h, \mathbf{S}); \mathbf{S} \in \mathfrak{A}\}. \tag{4}$$

The following properties of the variation are fundamental and easily proved.

LEMMA 1. *If E is an elementary set and $\mathbf{S}^* \subseteq \mathbf{S}(E) \cup \mathbf{S}(\backslash E)(\mathbf{S}, \mathbf{S}^* \subseteq \mathfrak{I})$ then $V(h, \mathbf{S}^*) \leq V(h, \mathbf{S}(E)) + V(h, \mathbf{S}(\backslash E)) \leq V(h, \mathbf{S})$.*

LEMMA 2. *If $(T, \mathfrak{A}, \mathfrak{I})$ is an additive division space then*

$$V(h, \mathfrak{A}) = V(h, \mathfrak{A}(E)) + V(h, \mathfrak{A}(\backslash E)) \tag{5}$$

for every elementary set E.

Here and elsewhere we define

$$\mathfrak{A}(X) = \{\mathbf{S}(X) : \mathbf{S} \in \mathfrak{A}\} \quad \textit{and} \quad \mathfrak{A}[X] = \{\mathbf{S}[X] : \mathbf{S} \in \mathfrak{A}\} \quad (X \subseteq T).$$

LEMMA 3. *If $(T, \mathfrak{A}, \mathfrak{I})$ is an additive division space and $\mathbf{S} \in \mathfrak{A}$ with*

$$V(h, \mathbf{S}) \leq V(h, \mathfrak{A}) + \varepsilon < +\infty$$

for some $\varepsilon > 0$, then $V(h, \mathbf{S}(E)) \leq V(h, \mathfrak{A}(E)) + \varepsilon$ for every elementary set E.

Proof. By Lemma 1

$$V(h, \mathbf{S}(E)) \leq V(h, \mathbf{S}) - V(h, \mathbf{S}(\backslash E) \leq V(h, \mathfrak{A}) + \varepsilon - V(h, \mathfrak{A}(\backslash E))$$

so that (5) gives the result. Note that Lemma 2 is the only use of the additivity hypothesis here.

The variation gives rise to a set function, an outer measure on T in most cases, which serves as the only contact the present theory need have with formal measure theory.

DEFINITION 2. If h is a function on $\mathfrak{I}$ with values in a normed linear space the h-variation, h^*, is defined by

(i) $V(h, \mathbf{S}; X) = V(h, \mathbf{S}[X])$ $(\mathbf{S} \subseteq \mathfrak{I}, X \subseteq T)$

(ii) $h^*(X) = V(h, \mathfrak{A}; X) = V(h, \mathfrak{A}[X])$ $(X \subseteq T)$.

Theorem 1 states the properties of the variation function which are the key points of the theory.

THEOREM 1. *Let* $(T, \mathfrak{A}, \mathfrak{I})$ *be a division space and* h *a function on* $\mathfrak{I}$ *with values in a normed linear space. Then*:

$$h^*(\phi) = 0 \quad \text{and} \quad 0 \leq h^*(X) \leq +\infty \quad (X \subseteq T); \tag{6}$$

$$h^*(X_1 \cup X_2) \leq h^*(X_1) + h^*(X_2) \quad (X_1, X_2 \subseteq T); \tag{7}$$

$$h^*(X_1) \leq h^*(X_2) \quad (X_1 \subseteq X_2 \subseteq T). \tag{8}$$

If $(T, \mathfrak{A}, \mathfrak{I})$ *is decomposable and* $\{X_j\}$ *is a sequence of subsets of* T *then*

$$h^*\left(\bigcup_{j=1}^{\infty} X_j\right) \leq \sum_{j=1}^{\infty} h^*(X_j). \tag{9}$$

If $(T, \mathfrak{A}, \mathfrak{I})$ *is decomposable and* $X_1, X_2, X_3, \ldots$ *is an increasing sequence of subsets of* T *with each* $h\chi_{X_j}$ *satisfying* (5) (*thus in particular if* $(T, \mathfrak{A}, \mathfrak{I})$ *is additive*), *then*

$$h^*\left(\bigcup_{j=1}^{\infty} X_j\right) = \lim_{j\to\infty} h^*(X_j). \tag{10}$$

Proof. The proofs of (6), (7), and (8) are elementary. For (9) let $\varepsilon > 0$ and for each index j choose $\mathbf{S}_j \in \mathfrak{A}$ so that

$$V(h, \mathbf{S}_j; X_j) \leq V(h, \mathfrak{A}; X_j) + \varepsilon/2^j.$$

Without loss of generality (use (8)) we may assume that the $\{X_j\}$ are disjoint and so take $\mathbf{S} \in \mathfrak{A}$ such that $\mathbf{S}[X_j] \subseteq \mathbf{S}_j[X_j]$ for each j.

Let $\mathfrak{D} \subseteq \mathbf{S}\,[\bigcup_{j=1}^{\infty} X_j]$ be an arbitrary division and set $\mathfrak{D}_j = \mathfrak{D}[X_j]$. Then

$$\begin{aligned}(\mathfrak{D}) \sum \|h(I, x)\| &= \sum_{j=1}^{\infty} (\mathfrak{D}_j) \sum \|h(I, x)\| \\ &\leq \sum_{j=1}^{\infty} V(h, \mathbf{S}_j; X_j) \\ &\leq \sum_{j=1}^{\infty} V(h, \mathfrak{A}; X_j)+\varepsilon\end{aligned}$$

and the result follows easily.

For (10) let $\varepsilon>0$ and again choose $\mathbf{S}_j \in \mathfrak{A}$ such that each

$$V(h, \mathbf{S}_j; X_j) \leq V(h, \mathfrak{A}; X_j)+\varepsilon/2^j$$

and choose $\mathbf{S} \in \mathfrak{A}$ so that $\mathbf{S}[X_j \backslash X_{j-1}] \subseteq \mathbf{S}_j[X_j \backslash X_{j-1}]$ for each j (set $X_0=\phi$).

Let $\mathfrak{D} \subseteq \mathbf{S}[\bigcup_{j=1}^{\infty} X_j]$ be an arbitrary division and write $\mathfrak{D}_j=\mathfrak{D}[X_j \backslash X_{j-1}]$, $E_j=\sigma(\mathfrak{D}_j)$ and let m be the least integer for which $\mathfrak{D}_j=\phi$ $(j \geq m)$. Then

$$\begin{aligned}(\mathfrak{D}) \sum \|h(I, x)\| &= \sum_{j=1}^{m} (\mathfrak{D}_j) \sum \|h(I, x)\| \\ &\leq \sum_{j=1}^{m} V(h, \mathbf{S}_j(E_j); X_j) \\ &\leq \sum_{j=1}^{m} V(h, \mathfrak{A}(E_j); X_j)+\varepsilon.\end{aligned}$$

This last inequality follows from Lemma 3 applied to the function $h\chi_{X_j}$ and the definition of the $\{\mathbf{S}_j\}$. But

$$\begin{aligned}\sum_{j=1}^{m} V(h, \mathfrak{A}(E_j); X_j) &\leq \sum_{j=1}^{m} V(h, \mathfrak{A}(E_j); X_m) \\ &\leq V(h, \mathfrak{A}; X_m).\end{aligned}$$

Thus it follows easily that

$$h^*\left(\bigcup_{j=1}^{\infty} X_j\right) \leq \lim_{m\to\infty} h^*(X_m)$$

which, with (8), proves the final statement of the theorem.

DEFINITION 3. (i) A function H defined on the elementary subsets of a division space $(T, \mathfrak{A}, \mathfrak{I})$ and with values in a normed linear space is said to be *additive* on $(T, \mathfrak{A}, \mathfrak{I})$ if $H(E)=(\mathfrak{D}) \sum H(I)$ for every division $\mathfrak{D}$ of E, and every elementary set E.

With no change in notation we permit H to be defined on $\mathfrak{I}$ by writing

$$H(I, x) = H(I)((I, x) \in \mathfrak{I}).$$

(ii) A function h on $\mathfrak{I}$ with values in a normed linear space $\mathbf{E}$ is said to be integrable on $(T, \mathfrak{A}, \mathfrak{I})$ if there is an $\mathbf{E}$-valued additive function H on $(T, \mathfrak{A}, \mathfrak{I})$ such that $V(H-h, \mathfrak{A})=0$. Such an H, if it exists, is unique and so we write $\int h=H$ and $\int_E h=H(E)$ (E elementary set). If in addition $V(h, \mathfrak{A})<+\infty$, h is said to be *summable*.

The definition of the above integral (technically the *strong variational integral* [4, p. 522]) is purely descriptive. For the Riemann-type definition and the relation between the two (see [3], [4]) and for a detailed discussion of the same on the real line see [2]. Note that our definition of the integral is a little stronger than that in [4], but it permits us to avoid the concept of "partial set" and in most cases of interest yields the same theory. McShane [5] chooses to develop the theory without alluding to the idea of variation and so there is not a large degree of interplay between that memoir and the present paper.

2. **The Lebesgue spaces.** Throughout this section $\mathbf{E}$, $\mathbf{F}$, and $\mathbf{G}$ will denote arbitrary but fixed normed linear spaces, real or complex, with the norm in each written as $|\cdot|$. We assume there is a bilinear mapping $u: \mathbf{E}\times\mathbf{F}\to\mathbf{G}$ satisfying $|u(x, y)| \le |x|\,|y|$ $(x\in\mathbf{E}, y\in\mathbf{F})$.

Let $\mathbf{m}$ be a function on $\mathfrak{I}$ with values in $\mathbf{F}$: $\mathfrak{L}_{\mathbf{E}}(\mathbf{m})$ is defined to be the linear space of all $\mathbf{E}$-valued functions f on T for which the function $f\mathbf{m}$: $(I, x)\to u(f(x), \mathbf{m}(I, x))$ is summable with $V(|f|\,|\mathbf{m}|, \mathfrak{A}) < +\infty$, and $\mathfrak{L}_{\mathbf{E}}(\mathbf{m})$ is equipped with the seminorm

$$f \to \|f\|_{\mathbf{m}} = V(|f|\,|\mathbf{m}|, \mathfrak{A}).$$

If $\mathbf{E}$ is simply the scalar field, then the subscript may be omitted from $\mathfrak{L}_{\mathbf{E}}(\mathbf{m})$.

The theory of these spaces can be developed in much the same manner as the usual Lebesgue theory. Here the decomposability of the division space, which has provided the key results in Theorem 1, plays the role usually attributed to the "countable additivity" hypothesis in the classical theory: this permits the proof of Lemma 4, showing the completeness of the spaces $\mathfrak{L}_{\mathbf{E}}(\mathbf{m})$. Note that all our results follow from the properties of the variation proved in Theorem 1. For other approaches which provide the same conclusions, see [2], [3], or [4].

THEOREM 2. *Let $(T, \mathfrak{A}, \mathfrak{I})$ be a decomposable division space and suppose $\{f_n\}$ is fundamental (i.e. is a Cauchy sequence) in $\mathfrak{L}_{\mathbf{E}}(\mathbf{m})$. Then there exists a subsequence $\{f_{n_k}\}$ which is fundamental at every point of T excepting a set of $\mathbf{m}$-variation zero.*

Proof. For every $\varepsilon > 0$ define the set

$$X_{mn}(\varepsilon) = \{x\in T;\ |f_m(x)-f_n(x)| > \varepsilon\}$$

and observe that

$$\mathbf{m}^*(X_{mn}(\varepsilon)) \le \frac{1}{\varepsilon} V(|f_n-f_m|\,|\mathbf{m}|, \mathfrak{A};\ X_{mn}(\varepsilon))$$

$$\le \frac{1}{\varepsilon}\|f_n-f_m\|_{\mathbf{m}}.$$

Choose an integer $N(\varepsilon)$ so that $\|f_n-f_m\|_{\mathbf{m}} < \varepsilon^2$ whenever $m, n \ge N(\varepsilon)$ and then for such m, n we have $\mathbf{m}^*(X_{mn}(\varepsilon)) < \varepsilon$. Let $n_1 < n_2 < n_3 < \cdots$ be an increasing sequence of integers satisfying $n_k \geq N(1/2^k)$ for each k and define the sets $X_k = X_{n_k n_{k+1}}(1/2^k)$, $Y_k = \bigcup_{i \geq k} X_i$, and $Y_0 = \bigcap_1^\infty Y_k$.

Then, if $x \in T \setminus Y_m$ we have

$$|f_{n_k}(x) - f_{n_{k+1}}(x)| < 1/2^k \quad (k \geq m)$$

and hence, in a finite number of steps,

$$|f_{n_k}(x) - f_{n_p}(x)| < \sum_{i=k}^{p} \frac{1}{2^i} < \frac{1}{2^{k-1}} \quad (p > k \geq m).$$

Thus $\{f_{n_k}\}$ is fundamental at every point of $T \setminus Y_m$ for each integer m and so also at every point of $T \setminus Y_0$: it remains only to show that Y_0 has **m**-variation zero. But using (8) and (9),

$$\mathbf{m}^*(Y_0) \leq \mathbf{m}^*(Y_m) \leq \sum_{k \geqq m} \mathbf{m}^*(X_k) \leq \sum_{k \geqq m} \frac{1}{2^k} = \frac{1}{2^{m-1}} \quad \text{for every integer } m,$$

and it follows that $\mathbf{m}^*(Y_0) = 0$, which proves the theorem.

As an immediate consequence we obtain

COROLLARY 1. *Let* $(T, \mathfrak{A}, \mathfrak{I})$ *be a decomposable division space, let* **E** *be a Banach space and suppose* $\{f_n\}$ *is fundamental in* $\mathfrak{L}_{\mathbf{E}}(\mathbf{m})$. *Then there is a subsequence* $\{f_{n_k}\}$ *which converges at every point of T excepting a set of* **m**-*variation zero.*

The crucial theorem in Lebesgue theory is now available. We require here an additive division space, but this restriction can be relaxed in some circumstances. Note that no additional assumption whatever is needed on **m**.

THEOREM 3. *Let* $(T, \mathfrak{A}, \mathfrak{I})$ *be an additive decomposable division space and suppose that* **E** *and* **G** *are Banach spaces. If the sequence* $\{f_n\}$ *is fundamental in* $\mathfrak{L}_{\mathbf{E}}(\mathbf{m})$ *it converges to a function in that space.*

Proof. Let $\{f_{n_k}\}$, $\{Y_m\}$, and Y_0 be constructed from $\{f_n\}$ as in the proof of Theorem 2 and define

$$f(x) = \lim_{k \to \infty} f_{n_k}(x) \quad (x \in T \setminus Y_0) \qquad \text{and} \qquad f(x) = 0 \quad (x \in Y_0).$$

Set $g_k = f_{n_k} - f$ for each index k and observe that $\{g_k\}$ is "fundamental" in the sense that $\lim_{j,k \to \infty} V(|g_k - g_j|\,|\mathbf{m}|, \mathfrak{A}) = 0$. From this it follows that

$$\nu(X) = \lim_{k \to \infty} V(|g_k|\,|\mathbf{m}|, \mathfrak{A}; X)$$

exists uniformly for $X \subseteq T$, and hence that the set function ν inherits property (10).

Define the sets $A_{ij} = \{x : |f_i(x)| \geq 1/j\}$ and note that, since by hypothesis $V(|f_i|\,|\mathbf{m}|, \mathfrak{A}) < +\infty$, each $V(\mathbf{m}, \mathfrak{A}; A_{ij}) < +\infty$. From the construction of the sets $\{Y_m\}$ and the definition of the sequence $\{g_k\}$ we obtain

$$V(|g_k|\,|\mathbf{m}|, \mathfrak{A}; A_{ij} \setminus Y_m) \leq 1/2^{k-1} V(\mathbf{m}, \mathfrak{A}; A_{ij}) \quad (k \geq m)$$

so that each

$$\nu(A_{ij}\setminus Y_m) = \lim_{k\to\infty} V(|g_k|\,|\mathbf{m}|, \mathfrak{A}; A_{ij}\setminus Y_m) = 0.$$

Applying (10) to ν we obtain

$$\nu(A_{ij}) = \nu(A_{ij}\setminus Y_0) = \lim_{m\to\infty} (A_{ij}\setminus Y_m) = 0.$$

A further application of (10) along with the finite subadditivity of ν shows $\nu(A_0)$ $=0$ $(A_0=\bigcup_{i=1}^{\infty}\bigcup_{j=1}^{\infty} A_{ij})$. Hence finally

$$\lim_{k\to\infty} V(|f_{n_k}-f|\,|m|, \mathfrak{A}) = \lim_{k\to\infty} V(|f_{n_k}-f|\,|m|, \mathfrak{A}: A_0)$$
$$= \nu(A_0) = 0.$$

Since $\{f_n\}$ is fundamental it then easily follows that $\lim_{n\to\infty} V(|f_n-f|\,|\mathbf{m}|, \mathfrak{A})=0$.

Define $F_n=\int f_n\mathbf{m}$: for each elementary set E then, since each F_n is additive,

$$|F_n(E)-F_m(E)| \le V(F_n-F_m, \mathfrak{A}) \le \|f_n-f_m\|_{\mathbf{m}}$$

and $\lim_{n\to\infty} F_n(E)=F(E)$ exists in the Banach space G.

Moreover F is clearly additive and satisfies $\lim_{n\to\infty} V(F_n-F, \mathfrak{A})=0$. Thus for all n,

$$V(F-f\mathbf{m}, \mathfrak{A}) \le V(F-F_n, \mathfrak{A})+V(F_n-f_n\mathbf{m}, \mathfrak{A})+V(|f_n-f|\,|\mathbf{m}|, \mathfrak{A}),$$

from which it follows that $V(F-f\mathbf{m}, \mathfrak{A})=0$ and hence that $F=\int f\mathbf{m}$.

Since also

$$V(|f|\,|\mathbf{m}|, \mathfrak{A}) = \lim_{n\to\infty} V(|f_n|\,|\mathbf{m}|, \mathfrak{A}) < +\infty,$$

we have that f belongs to $\mathfrak{L}_{\mathbf{E}}$ $(\mathbf{m})$ and, as

$$\|f_n-f\|_{\mathbf{m}} = V(|f_n-f|\,|\mathbf{m}|, \mathfrak{A}) \to 0 \quad \text{as } n\to\infty,$$

$\{f_n\}$ converges to f in the space $\mathfrak{L}_{\mathbf{E}}(\mathbf{m})$, which completes the proof of the theorem.

Note that the only use made of the additivity of the division space was an application of (10) to the function $|g_k|\,|\mathbf{m}|$.

The remainder of this section follows the usual exposition of the Lebesgue theory, permitting the proof of the "dominated convergence theorem" to rest on the completeness of the space $\mathfrak{L}_{\mathbf{E}}(\mathbf{m})$. In our context this can be stated in a more revealing, though not more general, form.

Definition 4. Let $\{h_n\}$ be a sequence of functions defined on $\mathfrak{J}$ and with values in a normed linear space. The *mixed variation* of the sequence is defined as

$$V(\{h_n\}, \mathbf{S}) = \sup\,(\mathfrak{D}) \sum \|h_{n(I,x)}(I, x)\| \quad (\mathbf{S} \subseteq \mathfrak{J})$$

where the supremum is with respect to all divisions $\mathfrak{D} \subseteq \mathbf{S}$ and all functions n on $\mathfrak{D}$ with positive integer values.

If $(T, \mathfrak{A}, \mathfrak{J})$ is a division space we define

$$V(\{h_n\}, \mathfrak{A}) = \inf\{V(\{h_n\}, \mathbf{S}) : \mathbf{S} \in \mathfrak{A}\}$$

and say that $\{h_n\}$ has finite mixed variation on $(T, \mathfrak{A}, \mathfrak{J})$ if this is finite.

LEMMA 4. *Let* $(T, \mathfrak{A}, \mathfrak{J})$ *be an additive decomposable division space, let* μ *be a nonnegative function on* $\mathfrak{J}$ *and let* $\{f_n\}$ *be a sequence of nonnegative functions in* $\mathfrak{L}(\mu)$ *such that* $\{f_n\mu\}$ *has finite mixed variation.*

(a) *If* $\{f_n\}$ *is an increasing sequence then the function* $f(x) = \lim_{n\to\infty} f_n(x)$ (*if this is finite, and zero otherwise*) *belongs to* $\mathfrak{L}(\mu)$ *and* $\{f_n\}$ *converges to* f *in* $\mathfrak{L}(\mu)$.

(b) *The function* $f(x) = \sup_n f_n(x)$ (*if this is finite, and zero otherwise*) *belongs to* $\mathfrak{L}(\mu)$ *and* $\|f\|_\mu \geq \sup_n \|f_n\|_\mu$.

Proof. The proofs are standard: in (a) one shows that $\{f_n\}$ is fundamental in $\mathfrak{L}(\mu)$ and applies Theorem 3. For (b) one combines (a) and integrability theorems for the maximum of functions [4, p. 525] or [2, pp. 43–46] in the usual way.

THEOREM 4. *Let* $(T, \mathfrak{A}, \mathfrak{J})$ *be an additive decomposable division space and let* $\mathbf{E}$ *and* $\mathbf{G}$ *be Banach spaces. Suppose that a sequence* $\{f_n\}$ *in* $\mathfrak{L}_{\mathbf{E}}(\mathbf{m})$ *converges at every point, excepting a set of* $\mathbf{m}$*-variation zero, to a function* f *and that* $\{|f_n|\ |\mathbf{m}|\}$ *has finite mixed variation where each* $|f_n|$ *belongs to* $\mathfrak{L}(|\mathbf{m}|)$. *Then* f *belongs to* $\mathfrak{L}_{\mathbf{E}}(\mathbf{m})$ *and* $\{f_n\}$ *converges to* f *in that space.*

Proof. As in [1, pp. 135–136] for example one shows that $\{f_n\}$ is fundamental and applies Theorem 3 to obtain the result.

Note that the condition here that $\{|f_n|\ |\mathbf{m}|\}$ have finite mixed variation is actually equivalent to the existence of a function F in $\mathfrak{L}(|\mathbf{m}|)$ satisfying $|f_n(x)| \leq F(x)$ everywhere in T excepting a set of $\mathbf{m}$-variation zero (simply take $F(x) = \sup_n |f_n(x)|$ if finite and zero otherwise and apply Lemma 4(b)).

3. **Integration in locally compact spaces.** Let T be a locally compact Hausdorff space: a division space $(T, \mathfrak{A}, \mathfrak{J})$ is defined as follows. Let $\mathfrak{C}$ be the ring generated by the compact subsets of T and let $\mathfrak{J}$ denote the collection of all pairs (I, x) ($I \in \mathfrak{C}$ and $x \in T$). If N is an arbitrary function which assigns to each point $x \in T$ a neighborhood $N(x)$ of x we shall write $\mathbf{S}_N = \{(I, x) \in \mathfrak{J};\ I \subseteq N(x)\}$. Then $(T, \mathfrak{A}, \mathfrak{J})$ is a fully decomposable division space where $\mathfrak{A}$ denotes the collection of all $\mathbf{S}_N$ for arbitrary neighborhood functions N.

However $(T, \mathfrak{A}, \mathfrak{J})$ is not in general additive so that if $\mathbf{E}$, $\mathbf{F}$, $\mathbf{G}$, and $\mathbf{m}$ are as in the previous section then $\mathfrak{L}_{\mathbf{E}}(\mathbf{m})$ may not satisfy Theorem 3. Let $\mathbf{m}$ be an $\mathbf{F}$-valued additive function on the ring $\mathfrak{C}$ then it is easy to show that every function h of the form $h(I, x) = f(x)\mathbf{m}(I)$ ($f(x) \in \mathbf{E}$) satisfies (5) and so the proof of Theorem 3 holds in this case without modification.

To obtain further results we must impose certain restrictions on $\mathbf{m}$.

DEFINITION 5. An **F**-valued function **m** on $\mathfrak{I}$ is said to be *V-regular* if for every compact set $K \subseteq T$ and every $\varepsilon > 0$ there is an open set $U \supseteq K$ such that $V(\mathbf{m}, \mathfrak{A}(U \backslash K)) < \varepsilon$.

DEFINITION 6. By a *simple* **E**-valued function we shall mean any function of the form $(\mathfrak{D}) \sum e_x \chi_I$ $(e_x \in \mathbf{E})$ where $\mathfrak{D}$ is a division. $\mathfrak{C}_{\mathbf{E}}(T)$ will denote the linear space of all simple **E**-valued functions and $\mathfrak{K}_{\mathbf{E}}(T)$ the linear space of all compactly supported continuous **E**-valued functions.

THEOREM 5. *Let* **m** *be an* **F**-*valued additive V-regular function such that* $\mathbf{m}^*$ *is finite on compact sets. Then* $\mathfrak{L}_{\mathbf{E}}(\mathbf{m})$ *contains* $\mathfrak{C}_{\mathbf{E}}(T)$.

Proof. Suppose $e \in \mathbf{E}$ and that $K \subseteq T$ is compact: we show that $f = e\chi_K$ is in $\mathfrak{L}_{\mathbf{E}}(\mathbf{m})$. The extension to an arbitrary element of $\mathfrak{C}_{\mathbf{E}}(\mathbf{m})$ then follows by linearity.

For every elementary set E $(E \in \mathfrak{C})$ define $H(E) = e\mathbf{m}\,(K \cap E)$ so that H is an additive **G**-valued function on $\mathfrak{C}$. For any $\varepsilon > 0$ choose a neighborhood U of K so that $V(\mathbf{m}, A(U \backslash K) < \varepsilon$ and a neighborhood function N so that $N(\backslash K) \cap K = \phi$ and $N(U) \subseteq U$. Then if $\mathfrak{D} \subseteq \mathbf{S}_N$ is a division

$$(\mathfrak{D}) \sum |H(I) - f(x)\mathbf{m}(I)| \le (\mathfrak{D}[K]) \sum |e|\, |\mathbf{m}(I \cap K) - \mathbf{m}(I)| + (\mathfrak{D}[\backslash K]) \sum |e|\, |\mathbf{m}(I \cap K)|.$$

By the construction of N the second sum must vanish; since **m** is additive we have

$$\mathbf{m}(I \backslash K) = \mathbf{m}(I) - \mathbf{m}(I \cap K)$$

and if $(I, x) \in \mathfrak{D}[K]$ then $(I \backslash K, x) \in \mathbf{S}_N(U \backslash K)$, so that

$$(\mathfrak{D}) \sum |H(I) - f(x)\mathbf{m}(I)| \le |e|\, V(\mathbf{m}, \mathbf{S}_N(U \backslash K)).$$

From this we can argue that

$$V(H - f\mathbf{m}, \mathfrak{A}) \le |e|\, V(\mathbf{m}, \mathfrak{A}(U \backslash K)) < |e|\, \varepsilon.$$

As $|e| < +\infty$ and $\varepsilon > 0$ is arbitrary this proves that $f\mathbf{m}$ is integrable. Also $V(|f|\, |\mathbf{m}|, \mathfrak{A}) \le |e|\, \mathbf{m}^*(K) < +\infty$ so that f belongs to $\mathfrak{L}_{\mathbf{E}}(\mathbf{m})$ as required which completes the proof of the theorem.

If **E** and **G** are Banach spaces and **m** is an **F**-valued additive V-regular function with $\mathbf{m}^*$ finite on compact sets then Theorems 3 and 5 lead in the usual manner to several useful results. In particular $\mathfrak{L}_{\mathbf{E}}(\mathbf{m})$ includes both $\mathfrak{C}_{\mathbf{E}}(T)$ and $\mathfrak{K}_{\mathbf{E}}(T)$, $\mathfrak{C}_{\mathbf{E}}(T)$ and $\mathfrak{K}_{\mathbf{E}}(T)$ have the same closures in $\mathfrak{L}_{\mathbf{E}}(\mathbf{m})$ ([5, p. 43] discusses the situation in which this is the whole of $\mathfrak{L}_{\mathbf{E}}(\mathbf{m})$) and both closures include functions of the form $\chi_E f$ $(f \in \mathfrak{K}_{\mathbf{E}}(T)$ and $\mathrm{E} \in \mathfrak{C})$. These statements will be used in the next section without further reference.

4. **Representation of operators.** Let T be a locally compact Hausdorff space and let $(T, \mathfrak{A}, \mathfrak{I})$ be the associated division space as constructed in the previous section.

Suppose that $\mathbf{E}$, $\mathbf{F}$, and $\mathbf{G}$ are Banach spaces with $\mathbf{F}=\mathfrak{L}(\mathbf{E},\mathbf{G})$ (the space of all continuous linear transformations from $\mathbf{E}$ into $\mathbf{G}$) and that $\mathfrak{K}_{\mathbf{E}}(T)$ is the linear space of all compactly supported continuous functions on T with values in $\mathbf{E}$. The problem discussed in the present section is that of representing certain linear operators from $\mathfrak{K}_{\mathbf{E}}(T)$ into $\mathbf{G}$ by an appropriate integral with respect to an $\mathbf{F}$-valued set function. This problem, motivated originally by the Riesz–Kakutani theorem, has received some attention in the literature (see [1, p. 416]). We show that a certain class of operators (those that are *dominated*), introduced to simplify the problem, serves to characterize in a sense the strong variational integral.

DEFINITION 7. Let Γ be a linear mapping from $\mathfrak{K}_{\mathbf{E}}(T)$ into $\mathbf{G}$. We define

$$|||\Gamma_A||| = \sup \sum_{k=1}^{n} |\Gamma(f_k)| \quad (A \subseteq T)$$

where $\{f_k: k=1,\dots,n\}$ denotes an arbitrary finite sequence of functions in $\mathfrak{K}\ (T)$ such that $|f_j(x)|\,|f_k(x)|=0\ (j\neq k)$ and $|f_k(x)|\leq\chi_A(x)$ for all $x\in T$.

If $A\to|||\Gamma_A|||$ is finite on compact sets then Γ is said to be *dominated* [1, p. 383].

THEOREM 6. *Let* $\mathbf{m}$ *be an additive V-regular function with values in* $\mathfrak{L}(\mathbf{E},\mathbf{G})$ *and which has finite variation on compact sets. Then* $M\colon f\to\int_{K(f)}f\mathbf{m}$ $(K(f)=\operatorname{supp} f)$ *is a dominated linear mapping from* $\mathfrak{K}_{\mathbf{E}}(T)$ *into* $\mathbf{G}$ *such that* $|||M_G|||\leq\mathbf{m}^*(G)$ *for every open set* G.

Proof. The integral is defined in the sense of §2 with $\mathbf{F}=\mathfrak{L}(\mathbf{E},\mathbf{G})$ and u as the canonical mapping from $\mathbf{E}\times\mathbf{F}$ into $\mathbf{G}$.

The remarks in §3 show that M is defined on $\mathfrak{K}_{\mathbf{E}}(T)$ and linearity is obvious; it is sufficient then to prove the inequality $\mathbf{m}^*(G)\geq|||M_G|||$ (G open) for then the fact that M is dominated follows from the assumed finiteness of $\mathbf{m}^*$ on compact sets.

To this end let G be open and let $\{f_k;\ k=1,2,\dots,n\}$ be a sequence of functions satisfying the conditions of Definition 7 with $A=G$. Set $G_k=f_k^{-1}(\backslash\{0\})$ so that $\{G_k\}$ is a sequence of disjoint relatively compact open sets; construct a neighborhood function N so that each $N(G_k)\subseteq G_k$ and set $f=\sum_{k=1}^{n}f_k$. Then

$$\begin{aligned} V(\mathbf{m},\mathbf{S}_N;G) &\geq V(|f|\,|\mathbf{m}|,\mathbf{S}_N;G)\\ &\geq \sum_{k=1}^{n} V(|f_k|\,|\mathbf{m}|,\mathbf{S}_N;G_k)\\ &\geq \sum_{k=1}^{n} |M(f_k)| \end{aligned}$$

and so $V(\mathbf{m},\mathbf{S}_N;G)\geq|||M_G|||$, from which the final assertion of the theorem now follows.

LEMMA 5. *If* $\Gamma\colon\mathfrak{K}_{\mathbf{E}}(T)\to\mathbf{G}$ *is a dominated linear mapping then the function*

$$\gamma(E) = \inf\{|||\Gamma_G|||;\ G \text{ open},\ E\subseteq G\}$$

defined for all elementary sets E is nonnegative, additive, and V-regular and has finite variation on compact sets. Moreover $|\Gamma(f)| \leq V(f\gamma, \mathfrak{A})$ *for every* $f \in \mathfrak{K}_{\mathbf{E}}(T)$ *and* $\gamma^*(G) \leq |||\Gamma_G|||$ *for every open set G.*

Proof. If we set $\gamma'(A) = \inf\{|||\Gamma_G|||;\, G \text{ open, } A \subseteq G\}$ for every set A ($A \subseteq T$) then the arguments of [1, pp. 383–387] can be applied to show that γ' is a regular Borel (outer) measure in the usual sense. Thus γ, which is the restriction of γ' to the ring $\mathfrak{E}$, can be shown to have the properties stated in the lemma. We also obtain that $\gamma^*(E) \leq \gamma(E)$ ($E \in \mathfrak{E}$): in fact if $G \supseteq E$, G open, and N is a neighborhood function with $N(G) \subseteq G$ then for every division $\mathfrak{D} \subseteq \mathbf{S}_N[E]$ we have $E' = \sigma(\mathfrak{D}) \subseteq G$ and so

$$(\mathfrak{D}) \sum \gamma(I) = \gamma(E') \leq |||\Gamma_G|||.$$

From this we obtain $\gamma^*(E) \leq |||\Gamma_G|||$ and so

$$\gamma^*(E) \leq \inf_{G \supseteq E} |||\Gamma_G||| = \gamma(E).$$

Let now $f \in \mathfrak{K}_{\mathbf{E}}(T)$ and $\varepsilon > 0$: we choose a neighborhood function N so that $|f(y) - f(x)| < \varepsilon$ ($y \in N(x)$) and let $\mathfrak{D} = \{(I_k, x_k);\, k = 1, 2, \ldots, n\} \subseteq \mathbf{S}_N$ be a division of $K = \operatorname{supp} f$, and G_k be arbitrary open sets with $I_k \subseteq G_k \subseteq N(x_k)$. Since the $\{G_k;\, k = 1, 2, \ldots, n\}$ cover the compact set $K(f)$ there are nonnegative continuous functions $\{\varphi_k;\, k = 1, 2, \ldots, n\}$ such that each φ_k vanishes outside G_k and $\sum_{k=1}^{n} \varphi_k(x) = 1$ ($x \in K$). Then

$$\begin{aligned}
|\Gamma(f)| &= \Gamma\left(\sum_{k=1}^{n} \varphi_k f\right) \leq \sum_{k=1}^{n} |\Gamma(\varphi_k f)| \\
&\leq \sum_{k=1}^{n} \sup_{t \in G_k} |f(t)|\, |||\Gamma_{G_k}||| \\
&\leq \sum_{k=1}^{n} [\,|f(x_k)|\, \gamma'(G_k) + \varepsilon \gamma'(G_k)].
\end{aligned}$$

If we now take the infimum of the right side of this inequality for all such choices of $\{G_k\}$ we obtain

$$\begin{aligned}
|\Gamma(f)| &\leq (\mathfrak{D}) \sum [\,|f(x)|\, \gamma(I) + \varepsilon\gamma(I)] \\
&\leq V(f\gamma, \mathbf{S}_N) + \varepsilon\gamma(K(f)).
\end{aligned}$$

As $\varepsilon > 0$ is arbitrary and γ is finite on compact sets it then follows that $|\Gamma(f)| \leq V(f\gamma, \mathfrak{A})$ as required.

For the final assertion of the lemma let G be open, let $\varepsilon > 0$, and choose a neighborhood function N so that $N(G) \subseteq G$. We choose a division $\mathfrak{D} \subseteq \mathbf{S}_N[G]$ so that

$$V(\gamma, \mathbf{S}_N; G) \leq (\mathfrak{D}) \sum \gamma(I) + \varepsilon.$$

Then if $E=\sigma(\mathfrak{D})$ we have $E \subseteq G$ and so

$$\gamma^*(G) \le V(\gamma, \mathbf{S}_N; G) \le \gamma(E)+\varepsilon \le |||\Gamma_G|||+\varepsilon.$$

As $\varepsilon>0$ is arbitrary this completes the proof of the lemma.

THEOREM 7. *A linear mapping* Γ *of* $\mathfrak{R}_{\mathbf{E}}(T)$ *into* $\mathbf{G}$ *is dominated if and only if there is an additive V-regular function* $\mathbf{m}$ *with values in* $\mathfrak{L}(\mathbf{E}, \mathbf{G})$ *and which has finite variation on compact sets such that*

$$\Gamma(f) = \int_{K(f)} f\mathbf{m} \quad (K(f) = \operatorname{supp} f).$$

Then necessarily $|||\Gamma_G|||=\mathbf{m}^*(G)$ *for every open set* G.

Proof. The sufficiency has already been established in Theorem 6. Conversely suppose Γ is dominated and construct the function γ as in Lemma 5. We must then have $\mathfrak{R}_{\mathbf{E}}(T) \subseteq \mathfrak{L}_{\mathbf{E}}(\gamma)$ and $|\Gamma(f)| \le V(f\gamma, \mathfrak{A}) = \|f\|_\gamma$ for every $f \in \mathfrak{R}_{\mathbf{E}}(T)$.

Now since $\mathbf{G}$ is complete we may extend Γ to a linear map Γ' on $\overline{\mathfrak{R}_{\mathbf{E}}(T)}$ (the closure of $\mathfrak{R}_{\mathbf{E}}(T)$ in $\mathfrak{L}_{\mathbf{E}}(\gamma)$) so that the inequality $|\Gamma'(f)| \le \|f\|_\gamma$ continues to hold.

For each elementary set E we define the linear mapping $\mathbf{m}(E)$ of $\mathbf{E}$ into $\mathbf{G}$ by putting $\mathbf{m}(E)e=\Gamma'(e\chi_E)$ $(e \in \mathbf{E})$. Since

$$\sup_{|e|\le 1} |\Gamma'(e\chi_E)| \le \sup_{|e|\le 1} V(e\chi_E\gamma, \mathfrak{A}) \le \gamma^*(E)$$

we have that $\mathbf{m}(E) \in \mathfrak{L}(\mathbf{E}, \mathbf{G})$ and $\|\mathbf{m}(E)\| \le \gamma^*(E)$ (where $\|\cdot\|$ denotes the canonical norm in $\mathfrak{L}(\mathbf{E}, \mathbf{G})$). Thus $\mathbf{m}$ is an additive $\mathfrak{L}(\mathbf{E}, \mathbf{G})$-valued function and the inequality $\|\mathbf{m}(E)\| \le \gamma^*(E) \le \gamma(E)$ shows that $\mathbf{m}$ has finite variation on compact sets and is V-regular as a result of the corresponding properties for γ.

Let $f \in \mathfrak{R}_{\mathbf{E}}(T)$, $\varepsilon>0$ and choose a neighborhood function N so that $|f(x)-f(y)|<\varepsilon$ $(y \in N(x))$ and so that $N(K(f)) \subseteq F$ where F is any compact set containing $\operatorname{supp} f$ in its interior. Set $H(E)=\Gamma'(f\chi_E)$ for each elementary set E: then if $\mathfrak{D} \subseteq \mathbf{S}_N$ is a division and $\mathfrak{D}'=\mathfrak{D}[K(f)]$

$$\begin{aligned}
(\mathfrak{D}) \sum |H(I)-f(x)\mathbf{m}(I)| &= (\mathfrak{D}) \sum |\Gamma'(f\chi_I)-\Gamma'(f(x)\chi_I)| \\
&\le (\mathfrak{D}) \sum V((f\chi_I-f(x)\chi_I)\gamma, \mathfrak{A}) \\
&\le (\mathfrak{D}') \sum \varepsilon\gamma^*(I) \\
&\le (\mathfrak{D}') \sum \varepsilon\gamma(I) \\
&\le \varepsilon\gamma(F)
\end{aligned}$$

As $\varepsilon>0$ is arbitrary and $\gamma(F)$ is finite we obtain finally $V(H-f\mathbf{m}, \mathfrak{A})=0$, so that $\int_{K(f)} f\mathbf{m}=H(K(f))=\Gamma(f)$ as required.

The final assertion of the theorem follows on combining the inequalities in Theorem 6 and Lemma 5 to obtain

$$|||\Gamma_G||| \leq \mathbf{m}^*(G) \leq \gamma^*(G) \leq |||\Gamma_G|||$$

for every open set G, which completes the proof of the theorem.

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