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OUTER MEASURES AND TOTAL VARIATION

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In this note we collect some observations on the outer measures ψ_f and ψ^f that have been introduced in [4] and which describe the total variation of the function f. These properties have direct applications to the study of the derivative and the relative derivative. For definitions and notation the reader is referred to [4].

The outer measure ψ^f in particular is closely related to what Saks [5, p. 228] has called the "strong variation" of a function. The connection is, however, not as straightforward as might be hoped. For example, if $f(x) = \sin 1/x$ for positive x and $f(x) = 0$ otherwise, and K is the compact set $\{0\} \cup \{2/n : n = 1, 2, 3, \ldots\}$ then one can compute that $\psi_f(K) < \psi^f(K) = 1$ but that f is neither VB nor VB_* on the set K; hence, the finiteness of the measures does not imply that the variation (in either sense) is bounded on that set. On the other hand, a function may be VBG_* on a set without the measures ψ^f and ψ_f being σ-finite on that set since they can assign infinite measure at a point. Within these restrictions some results are, nonetheless, available.

THEOREM 1. *If f is VBG [respectively, VBG_*] on a set X then there is a set $C \subseteq X$ that is at most countable such that on $X \setminus C$ the measure ψ_f [respectively, the measure ψ^f] is σ-finite. Should f be bounded in some neighbourhood of every point in X then C may be taken empty.*

Proof. Suppose firstly that f is VBG on X so that there is a sequence of sets $\{E_k\}$ on each of which f is VB and such that $X \subseteq \cup E_k$. Let C be the collection of all points that are isolated in any set E_k and write $E'_k = E_k \setminus C$. Consider the collection

$$\mathbf{S}_k = \{([x, y], z) : x, y \in E'_k,\ z = x \text{ or } y\}.$$

Because E'_k contains no isolated points we can check that $\mathbf{S}_k$ is a member of $\mathfrak{H}_v[E'_k]$; because f is VB on E_k we must then have $V(f, \mathbf{S}_k) < +\infty$ and this shows that $\psi_f(E'_k)$ is finite for each k. Since $X \setminus C \subseteq \cup E'_k$ we have that ψ_f is σ-finite on $X \setminus C$ as required. The set C is certainly countable, and can be dropped if there is some assurance that ψ_f does not assign infinite measure to singletons, and so the theorem follows for the measure ψ_f.

For the measure ψ^f we prove the following lemma.

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LEMMA. *Suppose that f is VB_* on a compact set K and that $a = \inf K$ and $b = \sup K$ are the "endpoints" of K; then*

$$\psi^f(K \cap (a, b)) < +\infty.$$

If f is bounded in some neighbourhood of the points a and b then $\psi^f(K) < +\infty$.

To prove the lemma let M be the upper bound of the sums $\sum O(f, I_k)$ taken for sequences of nonoverlapping intervals $\{I_k\}$ with endpoints in K; the fact that f is VB_* on K says precisely that M is finite. Choose any $\mathbf{S} \in \mathfrak{H}[K \cap (a, b)]$ in such a way that every $(I, x) \in \mathbf{S}$ must have $I \subseteq (a, b)$ and let $\{(I_i, x_i)\}$ be any sequence from $\mathbf{S}$ that has nonoverlapping intervals $\{I_i\}$; define the numbers

$$[a_i, b_i] = I_i, \qquad \hat{a}_i = \sup K \cap [a, a_i], \quad \text{and} \quad \hat{b}_i = \inf K \cap [b_i, b]$$

all of which certainly exist since K is compact. Note that

$$|f(b_i) - f(a_i)| \leq O(f, [\hat{a}_i, x_i]) + O(f, [x_i, \hat{b}_i]$$

and that the collections $\{[\hat{a}_i, x_i]\}$ and $\{[x_i, \hat{b}_i]\}$ are each nonoverlapping with endpoints in K. (Degenerate intervals, of which there will, of course, be many, we will delete.) Accordingly then, $\sum |f(b_i) - f(a_i)|$ cannot exceed $2M$ so that $V(f, \mathbf{S})$ is finite, proving the lemma.

To return to the proof of the theorem, let f be VGB_* on X so that there must exist a sequence $\{K_n\}$ of compact sets on each of which f is VB_* and such that $X \subseteq \cup K_n$. Let C be the collection of the endpoints of any $\{K_n\}$ and apply the lemma to obtain $X \setminus C \subseteq \cup K_n \setminus C$ with ψ^f finite on each set $K_n \setminus C$ as required.

In the converse direction Henstock [3] has recently shown that the vanishing of ψ^f on a set X requires f to be continuous and VBG_* on that set. This same proof can be used to prove that if the outer measure ψ^f is σ-finite on a set X, then f must be VBG_* on X. The corresponding statement using the outer measure ψ_f and the concept VBG is false as Theorem 3 below shows.

We turn now to the question of the relationship between the values $\psi^f(X)$, $\psi_f(X)$, and $|f(X)|$ where (as in Saks [5]) $|E|$ denotes the Lebesgue outer measure of an arbitrary set E of real numbers and $f(X)$ is the image under f of a set X.

THEOREM 2. *Suppose that f is continuous on $\mathbf{R}$. Then $|f(X)| = 0$ implies that $\psi_f(X) = 0$. If f has, in addition, locally bounded variation on $\mathbf{R}$, then the converse is also true.*

Proof. Suppose to begin with that $|f(X)| = 0$; we follow an argument similar to Bruckner [2] to prove that $\psi_f(X) = 0$. Without loss of generality we may suppose that X is bounded and even that $X \subseteq [0, 1)$. Define now the following subsets of that interval:

M = the set of points in $[0, 1)$ at which f attains a strict relative maximum or minimum; I_{jn} = the interval $[(j-1)2^{-n}, j2^{-n})$ for any $n = 1, 2, \ldots$ and any $j = 1, 2, 3, \ldots, 2^n$; B_{jn} = the set of points x in $X \cap I_{jn}$ such that $f(x') = f(x)$ for some other point $x' \neq x$ in I_{jn}; $B_n = \bigcup_{j=1}^{2^n} B_{jn}$; $B = \cap B_n$; $D_n = X \setminus (M \cup B_n)$; and $D = \cup D_n$.

By our construction we have $X \subseteq D_n \cup M \cup B_n$ for every index n and thus $X \subseteq D \cup M \cup B_n$, which finally gives $X \subseteq D \cup M \cup B$. Thus in order to prove that $\psi_f(X) = 0$ we need only show that each of the sets D, M, and B has ψ_f-measure zero. M is immediate since it is denumerable (see Saks [5, p. 261]). If we show that for every index n, $\psi_f(D_n) = 0$, then $\psi_f(D)$ must vanish since ψ_f is an outer measure. This, in turn, will follow if we show that $\psi_f(D_n \cap I_{jn}) = 0$ for every index j and n. As argued in Bruckner [1] we can establish that the function f is monotonic on each $D_n \cap I_{jn}$; but since its values lie in a set of Lebesgue measure zero, it is straightforward to argue $\psi_f(D_n \cap I_{jn})$ must vanish.

There remains only to prove that $\psi_f(B) = 0$; observe that every point x in this set must have a sequence $\{x_n\}$ of points converging to x with the property that $f(x_n) = f(x)$ at every member of the sequence. Accordingly, there is an element $\mathbf{H}^*$ of $\mathfrak{H}_v[B]$ such that for every $(I, x) \in \mathbf{H}^*$, $f(I) = 0$. This immediately gives $\psi_f(B) = 0$ and completes the proof of the first part of the theorem.

For the second part of the theorem we need to prove that $\psi_f(X) = 0$ implies that $|f(X)| = 0$ under the additional assumption that f has locally bounded variation. We cite the following lemma:

LEMMA (Saks–Sierpiński). *For any bounded set $X \subseteq \mathbf{R}$ and any continuous monotonic function f on $\mathbf{R}$, $2\psi_f(X) \geq |f(X)|$.*

This is proved, in different language, of course, in Saks [5, p. 211].

The theorem now follows from this estimate if f is continuous and monotonic, and is readily extended to any continuous function that has locally bounded variation by replacing f by its variation function which is then both continuous and monotonic, and yields the same measure. This completes the proof.

Incidentally, we might remark that the vanishing of $\psi_f(X)$ whenever f maps x into a set of measure zero permits an improvement of [4, Theorem 5]. Applying Saks [5, p. 273], we must now have that for continuous f and g, with ψ^f being σ-finite on a set X, the extreme derivatives $\underline{f}_g(x)$ and $\overline{f}_g(x)$ must be equal ψ_g-almost everywhere in X.

THEOREM 8. *If f is continuous and has locally bounded variation on $\mathbf{R}$, then the measures ψ_f and ψ^f are identical.*

Proof. We may consider without loss of generality that f is continuous and monotonic since f could be replaced by its total variation function on any interval. If $X \subseteq \mathbf{R}$ and $\mathbf{H}^* \in \mathfrak{H}_v[X]$, then the family $\{I : (I, x) \in \mathbf{H}^*\}$ is a cover of X in the classical vitali sense; thus applying Vitali's theorem for the Lebesgue–Stieltjes measure μ_f associated with f there must exist a sequence $\{(I_i, x_i)\} \subseteq \mathbf{H}^*$ with nonoverlapping intervals such that

$$\mu_f(X) \leq \sum \mu_f(I_i) \leq \sum f(I_i) \leq V(f, \mathbf{H}^*).$$

But $\mu_f(X) \geq \psi^f(X)$ so that we have proved that $\psi^f(X) \leq \psi_f(X) \leq \psi^f(X)$ which establishes the theorem.

Thus for functions that are locally of bounded variation the outer measures ψ_f and ψ^f coincide with the usual Lebesgue–Stieltjes measure associated with such a function, at least in the event that f is continuous. For discontinuous functions [4, Theorem 4] illustrates the fact that the measures necessarily differ. It is possible to relax the hypotheses and prove this result for continuous functions that are VBG_* on the real line but it is doubtful that this can be extended any further. For example, it is easy to see that if every level set of f is perfect, then the measure ψ_f must necessarily vanish whereas the vanishing of ψ^f requires f to be constant. The extreme case is illustrated by our next theorem. Let us restrict attention to a particular interval $[a, b]$ and consider the outer measures ψ_f and ψ^f as arising from a function defined on this interval. By $C[a, b]$ we shall mean the usual Banach space of continuous functions on $[a, b]$ furnished with the supremum norm.

THEOREM 3. *For every function f in $C[a, b]$ excepting a subset of the first category in that space the measure ψ_f vanishes and the measure ψ^f is non σ-finite on each subinterval of $[a, b]$.*

Proof. We obtain the proof as an application of the derivation estimates of [4, Theorem 5] in conjunction with the known behaviour of typical functions in $C[a, b]$. Bruckner [2] gives a proof that, excepting a first category subset of $C[a, b]$, every function f has each extended real value as a derived number at each point, i.e., that the limit points of $[f(x+h)-f(x)]/h$ as $h \to 0+$ and $h \to 0-$ include every real number and $+\infty$ and $-\infty$.

For such a function f let $g(x) = x$ and apply [4, Theorem 5] to the derivatives $\underline{f}_g(x)$ and $\bar{f}_g(x)$; then ψ^f must be non σ-finite or the theorem is contradicted. Similarly, applying that theorem to the derivatives $\underline{g}_f(x)$ and $\bar{g}_f(x)$, we prove that $\psi_f([a, b]) = 0$ and the theorem is proved.

References

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