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A FULL DESCRIPTIVE DEFINITION OF THE GAGE INTEGRAL

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ABSTRACT. We consider a specific Riemann type integral, called the *gage integral*. Using variational measures, we characterize all additive functions of intervals that are indefinite gage integrals. The characterization generalizes the descriptive definition of the classical Denjoy-Perron integral to all dimensions.

The gage integral, introduced in [6, Chapter 11], is a Riemann type integral which extends, in a natural way, the most important properties of the classical Denjoy-Perron integral to higher dimensions. In particular, it integrates the partial derivatives of differentiable functions so that the usual Gauss-Green formula is satisfied. Our main result (Theorem 4.4) gives a complete characterization of additive functions of intervals that are indefinite gage integrals. The well-known characterization in dimension one ([7, Chapter VIII]) depends on the order structure of the real line, and does not permit a direct extension to a multidimensional situation.

We employ suitably modified techniques of variational measures (see [8]), and rely heavily on the Ward theorem (Theorem 3.1). Thus any generalization of our results to Riemann type integrals defined by means of figures or sets of finite perimeter may require an appropriate generalization of Ward's theorem.

The paper is organized as follows. After some necessary preliminaries in Section 1, we give a detailed motivation for the gage integral in Section 2. The precise definition of the gage integral is given; for its properties, however, we refer to [6]. In addition, the reader is assumed to have some familiarity with the one-dimensional Henstock-Kurzweil integral. Section 3 is devoted to variational measures, whose application to the gage integral in Section 4 yields the desired descriptive definition. As a corollary, we obtain a condition under which the variational measure associated to the flux of a continuous vector field is absolutely continuous and σ-finite.

1. **Preliminaries.** The set of all real numbers is denoted by $\mathbf{R}$, and the ambient space of this paper is $\mathbf{R}^m$ where m is a positive integer. The metric in $\mathbf{R}^m$ is induced by the maximum norm. For a set $E \subset \mathbf{R}^m$, we denote by ∂E, $d(E)$, and $|E|$ the boundary, diameter, and Lebesgue measure of E, respectively. The words "measure" and "measurable" as well as the expressions "almost all" and "almost everywhere" always refer to the Lebesgue

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measure in $\mathbf{R}^m$. As usual, a set $E \subset \mathbf{R}^m$ is called *negligible* whenever $|E| = 0$. We say subsets A and B of $\mathbf{R}^m$ *overlap* if their intersection is not negligible.

The $(m-1)$-dimensional Hausdorff measure in $\mathbf{R}^m$ is denoted by $\mathcal{H}$, and a set $T \subset \mathbf{R}^m$ of σ-finite measure $\mathcal{H}$ is called *thin*. We note that by definition the 0-dimensional Hausdorff measure is the counting measure, and hence thin subsets of $\mathbf{R}$ coincide with countable sets.

Unless stated otherwise, a function is always real-valued. A nonnegative function δ on a set $E \subset \mathbf{R}^m$ is called a *gage* or an *essential gage* (abbreviated as *e-gage*) on E whenever its *null set* $N_\delta = \{x \in E : \delta(x) = 0\}$ is thin or negligible, respectively.

A *cell* is a compact nondegenerate interval, and a *figure* is a finite, possibly empty, union of cells. If $A \subset \mathbf{R}^m$ is a figure, the numbers

$$\|A\| = \mathcal{H}(\partial A) \quad \text{and} \quad s(A) = \frac{|A|}{[d(A)]^m}$$

are called the *perimeter* and *shape* of A, respectively.

A *partition* is a collection $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ where $A_1, \dots, A_p$ are nonoverlapping cells, and $x_i \in A_i$ for $i = 1, \dots, p$. Given $E \subset \mathbf{R}^m$, we say P is

- a partition *in* E if $\bigcup_{i=1}^p A_i \subset E$,
- a partition *of* E if $\bigcup_{i=1}^p A_i = E$,
- *anchored* in E if $\{x_1, \dots, x_p\} \subset E$.

If $s(A_i) > \varepsilon$ for an $\varepsilon > 0$ and $i = 1, \dots, p$, we say that P is *ε-shapely*; when $\varepsilon < 1$, every partition in dimension one is ε-shapely. For a nonnegative function δ on E, a partition P anchored in E is called *δ-fine* whenever $d(A_i) < \delta(x_i)$ for $i = 1, \dots, p$.

2. **The gage integral.** To motivate the definition of the gage integral, we begin with a modified definition of the classical Henstock-Kurzweil integral (see [6, Section 6.1]), referred to as the HK-integral. Recall that a function F on a cell $A \subset \mathbf{R}$ defines an additive function on the family of all subcells of A by setting $F([c,d]) = F(d) - F(c)$ for each cell $[c,d] \subset A$. Moreover, this function of cells has a unique extension, still denoted by F, to an additive function on the family of all subfigures of A.

DEFINITION 2.1. A function f is HK-*integrable* on a cell $A \subset \mathbf{R}$ if there is a function F on A satisfying the following condition: given $\varepsilon > 0$, there is a positive function δ on A such that

$$\sum_{i=1}^{p} \left| f(x_i)|A_i| - F(A_i) \right| < \varepsilon$$

for each δ-fine partition $\{(A_1, x_1), \dots, (A_p, x_p)\}$ in A.

The equivalence of Definition 2.1 and the original definition of Henstock and Kurzweil ([6, Definition 6.1.1]) follows immediately from Henstock's lemma ([6, Lemma 2.3.1 and Proposition 6.1.5]). It is also clear that F, viewed as an additive function of cells, is the *indefinite* HK-*integral* of f. Moreover, a moment's reflection reveals that F, as a function of points, is continuous on A. Using this it is easy to prove the following proposition.

PROPOSITION 2.2. *A function f is* HK*-integrable on a cell $A \subset \mathbf{R}$ if and only if there is a continuous function F on A satisfying the following condition: given $\varepsilon > 0$, there is a gage δ on A such that*

$$\sum_{i=1}^{p} \left| f(x_i)|A_i| - F(A_i) \right| < \varepsilon$$

for each δ-fine partition $\{(A_1, x_1), \dots, (A_p, x_p)\}$ in A. The function F, viewed as an additive function of cells, is the indefinite integral of f.

PROOF. As the converse is obvious, assume the condition of the proposition is satisfied. Choose an $\varepsilon > 0$ and find a corresponding gage δ on A with null set $N_\delta = \{z_1, z_2, \dots\}$. Define a positive function Δ on A so that $\Delta(x) = \delta(x)$ for each $x \in A - N_\delta$ and

$$\left| f(z_n)|B| - F(B) \right| < \varepsilon 2^{-n}$$

for $n = 1, 2, \dots$ and each cell $B \subset A$ with $z_n \in B$ and $d(B) < \Delta(z_n)$. Such a choice of Δ is possible since F is a continuous function. Now for a Δ-fine partition $\{(A_1, x_1), \dots, (A_p, x_p)\}$ in A, we obtain

$$\begin{aligned} \sum_{i=1}^{p} \left| f(x_i)|A_i| - F(A_i) \right| &= \sum_{x_i \notin N_\delta} \left| f(x_i)|A_i| - F(A_i) \right| + \sum_{n=1}^{\infty} \sum_{x_i = z_n} \left| f(x_i)|A_i| - F(A_i) \right| \\ &\leq \varepsilon + 2\varepsilon \sum_{n=1}^{\infty} 2^{-n} = 3\varepsilon, \end{aligned}$$

and the proposition is proved.

A direct generalization of the HK-integral to higher dimensions is straightforward but disappointing: the most attractive feature of the one-dimensional HK-integral, its ability to integrate derivatives, is lost in the process. It was first recognized by Mawhin (see [5]) that in order to integrate partial derivatives of differentiable functions, we must employ ε-shapely partitions. However, doing this simplistically leads to an integral without the usual additive properties. Since Mawhin's work several authors tried to reinstate the additivity by inventing highly technical, often artificial, variations of the multidimensional HK-integral. In this section, we show that Proposition 2.2 leads to a very natural higher dimensional generalization of the HK-integral called the *gage integral.* The gage integral has good additive properties and still integrates partial derivatives.

The only concept in Proposition 2.2 whose generalization to higher dimensions is not obvious is that of the continuous function F. Indeed, a continuous function on a one-dimensional cell must be replaced by a *continuous additive function* on the family of all subfigures of a multidimensional cell. The elementary observation below paves the way.

OBSERVATION 2.3. A function F on a cell $A \subset \mathbf{R}$ is continuous if and only if the associated function of figures has the following property: given $\varepsilon > 0$, there is an $\eta > 0$ such that $|F(B)| < \varepsilon$ for each figure $B \subset A$ with $\|B\| < 1/\varepsilon$ and $|B| < \eta$.

The condition stated in Observation 2.3 remains meaningful for additive functions defined on subfigures of a cell $A \subset \mathbf{R}^m$, and thus suggests our next definition.

DEFINITION 2.4. An additive function F on the family of all subfigures of a cell $A \subset \mathbf{R}^m$ is called *continuous* whenever given $\varepsilon > 0$, there is an $\eta > 0$ such that $|F(B)| < \varepsilon$ for each figure $B \subset A$ with $\|B\| < 1/\varepsilon$ and $|B| < \eta$.

To further motivate Definition 2.4, we describe the smallest topology τ on the family $\mathcal{A}$ of all subfigures of a cell $A \subset \mathbf{R}^m$ such that each continuous additive function on $\mathcal{A}$ is τ-continuous. There is a metric ρ on $\mathcal{A}$ defined by

$$\rho(B, C) = |(B - C) \cup (C - B)|$$

for all figures $B, C \subset A$, and an additive function F on $\mathcal{A}$ is ρ-continuous if and only if it is absolutely continuous. For $n = 1, 2, \ldots$, we let $\mathcal{A}_n = \{B \in \mathcal{A} : \|B\| \leq n\}$, and define τ as the largest topology on $\mathcal{A}$ for which all imbeddings $(\mathcal{A}_n, \rho) \hookrightarrow (\mathcal{A}, \tau)$ are continuous. It is easy to verify that τ is a sequential topology induced by a nonmetrizable uniformity. With some effort one can show that the completion of the space $(\mathcal{A}, \tau)$ consists of all BV subsets of A (see [3, Chapter 5]).

DEFINITION 2.5. A function f is *gage integrable* on a cell $A \subset \mathbf{R}^m$ if there is a continuous additive function F defined on the family of of all subfigures of A satisfying the following condition: given $\varepsilon > 0$, there is a gage δ on A such that

$$\sum_{i=1}^{p} \left| f(x_i)|A_i| - F(A_i) \right| < \varepsilon$$

for each ε-shapely δ-fine partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ in A.

According to [6, Theorem 11.4.9], the gage integrability (abbreviated as g-integrability) of the above definition coincides with that defined in [6, Definition 11.4.1]. In particular, the function F is uniquely determined by f, and we call it the *indefinite g-integral* of f in A.

Every Lebesgue integrable function on a cell is g-integrable and the two indefinite integrals coincide ([6, Theorem 11.4.5]). It follows that if two functions on a cell are equal almost everywhere, then the g-integrability of one implies that of the other, and their integrals are equal ([6, Corollary 11.4.7]). Using this fact, we can and will, in the obvious way, introduce g-integrability and the g-integral for extended real-valued functions defined almost everywhere in a cell.

For additional properties of the g-integral, including the additivity and g-integrability of partial derivatives, we refer to [6, Chapter 11] (see also [6, Theorem 12.8.5]).

3. **Critical variation.** Let F be an additive function on the family of all subcells of a cell $A \subset \mathbf{R}^m$, and let $x \in A$. The *lower derivate* of F at x is the extended real number

$$\underline{F}(x) = \inf_{\alpha>0} \sup_{\beta>0} \left[\inf \frac{F(B)}{|B|} \right]$$

where the infimum in the brackets is taken over all cells $B \subset A$ with $x \in B$, $d(B) < \beta$, and $s(B) > \alpha$. The extended real-valued function $x \longmapsto \underline{F}(x)$ on A is denoted by $\underline{F}$. When $\underline{F}(x) = -\underline{(-F)}(x)$ is a real number, we say that F is *derivable* at x, and call this common value the *derivate* of F at x, denoted by $F'(x)$. By F' we denote the function $x \longmapsto F'(x)$ defined on the set of all $x \in A$ at which F is derivable. Note that our concept of derivation coincides with the *ordinary derivation* introduced in [7, Chapter IV, Section 2],

The pivotal result for our exposition is the theorem of Ward, whose proof can be found in [7, Chapter IV, Section 11].

THEOREM 3.1 (WARD). *Let F be an additive function on the family of all subcells of a cell $A \subset \mathbf{R}^m$, and let $E = \{x \in A : \underline{F}(x) > -\infty\}$. Then F is derivable at almost all $x \in E$.*

Let F be again an additive function on the family of all subcells of a cell $A \subset \mathbf{R}^m$. Given a set $E \subset A$, $\alpha > 0$ and a nonnegative function δ on E, let

$$F^{\alpha,\delta}(E) = \sup \sum_{i=1}^{p} |F(A_i)|$$

where the supremum is taken over all partitions $\{(A_1, x_1), \dots, (A_p, x_p)\}$ in A anchored in E that are δ-fine and α-shapely. The *critical variation* or the *essential critical variation* of F on E is the extended real number

$$\sup_{\alpha>0} \inf_{\delta} F^{\alpha,\delta}(E)$$

where the infimum is taken over all gages or essential gages δ on E, respectively; it is denoted by $F^*(E)$ or $F^{e*}(E)$, respectively. An easy verification reveals that the extended real-valued functions

$$F^*: E \longmapsto F^*(E) \quad \text{and} \quad F^{e*}: E \longmapsto F^{e*}(E)$$

are metric measures in A (see [4, Chapter 1]), and that the measure F^{e*} is absolutely continuous with respect to the Lebesgue measure. As each gage is an e-gage, we have $F^{e*} \leq F^*$.

DEFINITION 3.2. An additive function F on the family of all subcells of a cell $A \subset \mathbf{R}^m$ is called BV_* or BV_{e*} whenever A is of σ-finite measure F^* or F^{e*}, respectively.

In dimension one the essential critical variation was studied in [2] and [1]. The one-dimensional version of the following theorem was proved in [1] by essentially the same techniques we use below.

THEOREM 3.3. *Let F be an additive function on the family of all subcells of a cell $A \subset \mathbf{R}^m$. Then, for any measurable set $E \subset A$, we have*

$$F^{e*}(E) = \int_E |\underline{F}(x)|\, dx$$

where the integral is the usual Lebesgue integral in $\mathbf{R}^m$.

PROOF. The Lebesgue integral $\int_A |\underline{F}(x)|\,dx$ exists (possibly equal to $+\infty$), since by [7, Chapter IV, Theorem 4.2], the extended real-valued function $\underline{F}$ is measurable on A.

Assume first that $\int_E |\underline{F}(x)|\,dx < F^{e*}(E)$ and find an $\alpha > 0$ so that

$$\int_E |\underline{F}(x)|\,dx < F^{\alpha,\delta}(E) \tag{1}$$

for every e-gage δ on E. According to our assumption, $\underline{F}$ is finite almost everywhere in E, and so by Ward's theorem, F is derivable almost everywhere in E. For each $x \in A$ let

$$f(x) = \begin{cases} F'(x) & \text{if } x \in E \text{ and } F \text{ is derivable at } x, \\ 0 & \text{otherwise.} \end{cases}$$

The indefinite Lebesgue integral $G = \int f(x)\,dx$ is an additive function on the family of all subcells of A that is derivable to f almost everywhere in A ([7, Chapter IV, Theorem 6.3]).

Choose an $\varepsilon > 0$ and define an e-gage δ on E as follows: if $x \in E$ is such that $F'(x) = G'(x) = f(x)$, find a $\delta(x) > 0$ so that $|F(B) - G(B)| < \varepsilon|B|$ for each cell $B \subset A$ with $s(B) > \alpha$, $x \in B$ and $d(B) < \delta(x)$; otherwise, let $\delta(x) = 0$. For each α-shapely partition $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ in A anchored in E that is δ-fine, we obtain

$$\begin{aligned} \sum_{i=1}^{p} |F(A_i)| &< \sum_{i=1}^{p} [\varepsilon|A_i| + |G(A_i)|] \le \sum_{i=1}^{p} [\varepsilon|A_i| + \int_{A_i} |f(x)|\,dx] \\ &\le \varepsilon|A| + \int_A |f(x)|\,dx = \varepsilon|A| + \int_E |\underline{F}(x)|\,dx. \end{aligned}$$

Consequently

$$F^{\alpha,\delta}(E) \le \varepsilon|A| + \int_E |\underline{F}(x)|\,dx,$$

and a contradiction to (1) follows from the arbitrariness of ε.

Conversely, assume $F^{e*}(E) < \int_E |\underline{F}(x)|\,dx$. Let $Y = \{x \in E : \underline{F}(x) = -\infty\}$ and let n be a positive integer. For each $x \in Y$ there is an $\alpha_x > 0$ such that given $\theta > 0$, we can find a cell $B \subset A$ with $x \in B$, $s(B) > \alpha_x$, $d(B) < \theta$, and $F(B) < -n|B|$. Fix a positive integer j, let $Y_j = \{x \in Y : \alpha_x > 1/j\}$ and find an e-gage δ on E so that $F^{1/j,\delta}(E) < \int_E |\underline{F}(x)|\,dx$. The family $\mathcal{B}$ of all cells $B \subset A$ such that $s(B) > 1/j$, $d(B) < \delta(x)$ for an $x \in B \cap Y_j$, and $F(B) < -n|B|$ is a Vitali cover of $Y_j - N_\delta$. Noting that N_δ is a negligible set and using Vitali's covering theorem ([7, Chapter IV, Theorem 3.1]), we can find a $(1/j)$-shapely partition $P = \{(B_1, y_1), \ldots, (B_q, y_q)\}$ in A anchored in E, such that P is δ-fine, $F(B_i) < -n|B_i|$ for $i = 1, \ldots, q$, and $\sum_{i=1}^{q} |B_i| > |Y_j|/2$. It follows that

$$|Y_j| < 2\sum_{i=1}^{q} |B_i| < \frac{2}{n}\sum_{i=1}^{q} |F(B_i)| \le \frac{2}{n} F^{1/j,\delta}(E),$$

and so Y_j is negligible by the arbitrariness of n. Since $Y = \bigcup_{j=1}^{\infty} Y_j$ is also negligible, F is derivable almost everywhere in E by Ward's theorem.

Applying Luzin's theorem ([7, Chapter III, Theorem 7.1]), construct an increasing sequence $\{E_n\}$ of closed subsets of E such that F' is defined and continuous on each E_n, and $|E - \bigcup_{n=1}^{\infty} E_n| = 0$. Select a set $Z = E_n$ so that

$$F^{e*}(E) < \int_Z |\underline{F}(x)|\,dx = \int_Z |F'(x)|\,dx,$$

and choose an $\varepsilon > 0$. There is a positive function σ on Z such that $|F'(x) - F'(y)| < \varepsilon$ for each $x, y \in Z$ with $d(\{x,y\}) < \sigma(x)$. Given $x \in Z$, find a $\beta_x > 0$ so that

$$\big|F(C) - F'(x)|C|\big| < \varepsilon|C|$$

for each cell $C \subset A$ of sufficiently small diameter for which $x \in C$ and $s(C) > \beta_x$. For $j = 1, 2, \ldots$, let $Z_j = \{x \in Z : \beta_x > 1/j\}$, and select a measurable set X_j so that $Z_j \subset X_j \subset Z$ and $|X_j| = |Z_j|$. Replacing X_j by $\bigcap_{i=j}^{\infty} X_i$, we may assume $\{X_j\}$ is an increasing sequence. Since $Z = \bigcup_{j=1}^{\infty} X_j$, there is a positive integer k such that

$$F^{e*}(E) < \int_{X_k} |F'(x)|\, dx.$$

Find an e-gage Δ on E so that

$$F^{1/k,\Delta}(E) < \int_{X_k} |F'(x)|\, dx. \tag{2}$$

With no loss of generality, we may assume $\Delta(x) \le \sigma(x)$ for each $x \in Z$. The family $\mathcal{C}$ of all cells $C \subset A$ with $s(C) > 1/k$ and such that $d(C) < \Delta(x)$ and

$$\big|F(C) - F'(x)|C|\big| < \varepsilon|C|$$

for an $x \in C \cap Z_k$ is a Vitali cover of $Z_k - N_\Delta$. As N_Δ is a negligible set, according to Vitali's covering theorem, there is a disjoint sequence $\{C_i\}$ in $\mathcal{C}$ such that

$$|X_k - \bigcup_{i=1}^{\infty} C_i| = |Z_k - \bigcup_{i=1}^{\infty} C_i| = 0.$$

For $i = 1, 2, \ldots$, select a $z_i \in C_i \cap Z_k$ so that $d(C_i) < \Delta(z_i) \le \sigma(z_i)$ and

$$\big|F(C_i) - F'(z_i)|C_i|\big| < \varepsilon|C_i|,$$

and observe that

$$\begin{aligned}\int_{X_k} |F'(x)|\, dx &\le \sum_{i=1}^{\infty} \int_{C_i \cap Z} |F'(x)|\, dx \le \sum_{i=1}^{\infty} \Big(\sup_{x \in C_i \cap Z} |F'(x)|\Big)|C_i| \\ &\le \sum_{i=1}^{\infty} \big(|F'(z_i)| + \varepsilon\big)|C_i| < \sum_{i=1}^{\infty} \big(|F(C_i)| + 2\varepsilon|C_i|\big) \\ &\le \sum_{i=1}^{\infty} |F(C_i)| + 2\varepsilon|A|.\end{aligned}$$

Since, for each positive integer s, the collection $\{(C_1, z_1), \ldots, (C_s, z_s)\}$ is a partition in A anchored in E that is Δ-fine, we see that $\sum_{i=1}^{\infty} |F(C_i)| \le F^{1/k,\Delta}(E)$. Thus

$$\int_{X_k} |F'(x)|\, dx \le F^{1/k,\Delta}(E) + 2\varepsilon|A|,$$

and a contradiction to (2) follows from the arbitrariness of ε.

COROLLARY 3.4. *An additive function F on the family of all subcells of a cell $A \subset \mathbf{R}^m$ is derivable almost everywhere in A if and only if it is* BV_{e*}.

PROOF. If $A = \bigcup_{n=1}^{\infty} E_n$ and $F^{e*}(E_n) < +\infty$ for $n = 1, 2, \ldots$, then it follows from Theorem 3.3 that $|\underline{F}(x)| < +\infty$ for almost all $x \in A$. By Ward's theorem F is derivable almost everywhere in A.

Conversely, assume F is derivable in $A - E_0$ where E_0 is a negligible set, and let $E_n = \{x \in A - E_0 : |F'(x)| < n\}$ for $n = 1, 2, \ldots$. Then $E = \bigcup_{n=0}^{\infty} E_n$ and, by Theorem 3.3, we have

$$F^{e*}(E_n) = \int_{E_n} |F'(x)|\, dx \leq n|E_n| < +\infty$$

for $n = 1, 2, \ldots$. Since $F^*(E_0) = 0$, the corollary is proved.

4. **Descriptive definition.** Let $A \subset \mathbf{R}^m$ be a cell. We say that an additive function on the family of all subcells of A is *continuous* whenever the unique extension of F to an additive function on the family of all subfigures of A is continuous in the sense of Definition 2.4. With this convention in mind, we proceed to characterize those continuous additive functions on the family of all subcells of A which are indefinite g-integrals.

PROPOSITION 4.1. *Let f be a g-integrable function on a cell $A \subset \mathbf{R}^m$. If F is the indefinite g-integral of f, then F is* BV_*.

PROOF. With no loss of generality we may assume that f is a real-valued function defined on the whole of A. Choose an $\alpha > 0$, and for $n = 1, 2, \ldots$, let $E_n = \{x \in A : |f(x)| < n\}$. According to Definition 2.5, there is a gage δ on A such that

$$\sum_{i=1}^{p} \left| f(x_i)|A_i| - F(A_i) \right| < 1$$

for each α-shapely δ-fine partition $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ in A. If P is anchored in E_n, then

$$\sum_{i=1}^{p} |F(A_i)| < \sum_{i=1}^{p} |f(x_i)| \cdot |A_i| + 1 \leq n|A| + 1,$$

and hence $F^{\alpha,\delta}(E_n) \leq n|A| + 1$. From the arbitrariness of α, we conclude

$$F^*(E_n) \leq n|A| + 1 < +\infty,$$

and as $A = \bigcup_{n=1}^{\infty} E_n$, the proposition is proved.

Since each BV_* function is a BV_{e*} function, Corollary 3.4 together with Proposition 4.1 imply that each indefinite g-integral is derivable almost everywhere. A stronger statement, provided by the next theorem, is proved independently.

THEOREM 4.2. *Let f be a g-integrable function on a cell $A \subset \mathbf{R}^m$, and let F be the indefinite g-integral of f. For almost all $x \in A$ the function F is derivable at x and $F'(x) = f(x)$.*

PROOF. We may assume f is a real-valued function. Denote by E the set of all $x \in A$ for which either F is not derivable at x or $F'(x) \neq f(x)$. Given $x \in E$, we can find an $\alpha_x > 0$ so that for each $\beta > 0$ there is a cell $B \subset A$ with $x \in B$, $s(B) > \alpha_x$, $d(B) < \beta$, and

$$\left|f(x)|B| - F(B)\right| \geq \alpha_x |B|.$$

Fix an integer $n \geq 2$, let $E_n = \{x \in E : \alpha_x \geq 1/n\}$ and choose an $\varepsilon > 0$. There is a gage δ on A such that

$$\sum_{i=1}^{p} \left|f(x_i)|A_i| - F(A_i)\right| < \frac{\varepsilon}{n}$$

for each $(1/n)$-shapely δ-fine partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ in A. The family $\mathcal{B}$ of all cells $B \subset A$ with $s(B) > 1/n$ and such that $d(B) < \delta(x)$ and

$$\left|f(x)|B| - F(B)\right| \geq \frac{1}{n}|B|$$

for an $x \in B \cap E_n$ is a Vitali cover of $E_n - N_\delta$. As N_δ is a negligible set, by Vitali's covering theorem, there is a disjoint sequence $\{B_k\}$ in $\mathcal{B}$ such that $|E_n - \bigcup_{k=1}^{\infty} B_k| = 0$. In each $B_k \cap E_n$ select a point x_k so that $d(B_k) < \delta(x_k)$ and

$$\left|f(x_k)|B_k| - F(B_k)\right| \geq \frac{1}{n}|B_k|.$$

Observe that for $p = 1, 2, \ldots$, the collection $\{(B_1, x_1), \ldots, (B_p, x_p)\}$ is a $(1/n)$-shapely δ-fine partition in A. Thus

$$\sum_{k=1}^{p} |B_k| \leq n \sum_{k=1}^{p} \left|f(x_k)|B_k| - F(B_k)\right| < \varepsilon$$

for all positive integers p, and consequently $|E_n| \leq \sum_{k=1}^{\infty} |B_k| \leq \varepsilon$. The arbitrariness of ε implies that E_n is negligible, and so is the set $E = \bigcup_{n=2}^{\infty} E_n$.

DEFINITION 4.3. A continuous additive function on the family of all subcells of a cell $A \subset \mathbf{R}^m$ is called AC_* whenever F^* is absolutely continuous with respect to the Lebesgue measure.

It is easy to see that the AC_* functions defined above coincide with those of [6, Definition 11.6.3]. We are now ready to provide a full descriptive definition of the g-integral.

THEOREM 4.4. *Let F be a continuous additive function on the family of all subcells of a cell $A \subset \mathbf{R}^m$. The following conditions are equivalent:*

1. *F is AC_* and BV_*;*
2. *F is AC_* and BV_{e*};*
3. *F' exists almost everywhere in A, and F is its indefinite g-integral.*

PROOF (2 $\Rightarrow$ 3). By Corollary 3.4, F is derivable almost everywhere in A. Denote by E the negligible set of all $x \in A$ at which F is not derivable, and let

$$f(x) = \begin{cases} F'(x) & \text{if } x \in A - E, \\ 0 & \text{if } x \in E. \end{cases}$$

It suffices to show F is the indefinite g-integral of f. To this end, choose an $\varepsilon > 0$, and find a gage α on E so that $\sum_{i=1}^{p} |F(A_i)| < \varepsilon/2$ for each ε-shapely partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ anchored in E that is α-fine; such an α exists, since F is AC_* and E is a negligible set. On $A - E$ there is a positive function β such that

$$\left| f(x)|B| - F(B) \right| < \frac{\varepsilon}{2|A|}|B|$$

for each $x \in A - E$ and each cell $B \subset A$ with $s(B) > \varepsilon$, $x \in B$ and $d(B) < \beta(x)$. Define a gage δ on A by setting

$$\delta(x) = \begin{cases} \alpha(x) & \text{if } x \in E, \\ \beta(x) & \text{if } x \in A - E, \end{cases}$$

and choose an ε-shapely δ-fine partition $\{(C_1, z_1), \ldots, (C_q, z_q)\}$ in A. Then

$$\sum_{i=1}^{q} \left| f(z_i)|C_i| - F(C_i) \right| < \sum_{z_i \in E} |F(C_i)| + \frac{\varepsilon}{2|A|} \sum_{z_i \notin E} |C_i| \leq \varepsilon,$$

which implies F is, indeed, the indefinite g-integral of f.

(3 $\Rightarrow$ 1). It follows from Proposition 4.1 that F is BV_*. To show F is also AC_*, select a negligible set $E \subset A$ and an $\varepsilon > 0$. Making E larger, we may assume F is derivable everywhere in $A - E$. Then F is the indefinite g-integral of the function f defined in the proof of (2 $\Rightarrow$ 3). Find a gage δ on A so that

$$\sum_{i=1}^{p} \left| f(x_i)|A_i| - F(A_i) \right| < \varepsilon$$

for each ε-shapely δ-fine partition $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ in A. If such a partition P is anchored in E, we obtain $\sum_{i=1}^{p} |F(A_i)| < \varepsilon$, and consequently $F^{\varepsilon,\delta}(E) \leq \varepsilon$. This and the arbitrariness of ε imply $F^*(E) = 0$.

As the remaining implication (1 $\Rightarrow$ 2) is obvious, the proof of the theorem is completed.

REMARK 4.5. From the one dimensional case $m = 1$ in Theorem 4.4 it follows that a function F is ACG_* on an interval in the sense of Saks [7] if and only if F is BV_* and AC_* in our sense. It can be proved too that a continuous function F is VBG_* on an interval in the sense of Saks if and only if F is BV_* in our sense.

Let $A \subset \mathbf{R}^m$ be a cell and let $v: A \longrightarrow \mathbf{R}^m$ be a continuous vector field. The *flux* of v, *i.e.*, the map $B \longmapsto \int_{\partial B} v \cdot n_B \, d\mathcal{H}$ where n_B denotes the unit exterior normal of a cell $B \subset A$, is an additive continuous function defined on the family of all subcells of A ([6, Propositions 8.3.2 and 11.2.8]). The divergence theorem for the g-integral ([6, Theorem 11.7.5]) yields immediately an interesting corollary of Theorem 4.4. As usual, we let $|x - y| = d(\{x, y\})$ for all $x, y \in \mathbf{R}^m$.

COROLLARY 4.6. *Let F be the flux of a continuous vector field v defined on a cell $A \subset \mathbf{R}^m$. If there is a thin set $T \subset A$ such that*

$$\limsup_{y \to x} \frac{|v(y) - v(x)|}{|y - x|} < +\infty$$

for each $x \in A - (T \cup \partial A)$, then F is AC_* *and* BV_*.

The following proposition and example illuminate the relationship between the critical variation and essential critical variation.

PROPOSITION 4.7. *Let F be an additive function on the family of all subcells of a cell $A \subset \mathbf{R}^m$. If $F^*(N) = 0$ for each negligible set $N \subset A$, then $F^* = F^{e*}$. In particular, $F^* = F^{e*}$ whenever F is* AC_*.

PROOF. Assume $F^{e*}(E) < F^*(E)$ for a set $E \subset A$, and find an $\alpha > 0$ and an e-gage ω on E so that $F^{\alpha,\omega}(E) < F^{\alpha,\delta}(E)$ for each gage δ on E. Since $N = N_\omega$ is a negligible set, given $\varepsilon > 0$, there is a gage δ_N on N such that $F^{\alpha,\delta_N}(N) < \varepsilon$. Define a gage δ on E by setting

$$\delta(x) = \begin{cases} \omega(x) & \text{if } x \in E - N, \\ \delta_N(x) & \text{if } x \in N, \end{cases}$$

and observe that

$$F^{\alpha,\delta}(E) \leq F^{\alpha,\omega}(E) + F^{\alpha,\delta_N}(N) < F^{\alpha,\omega}(E) + \varepsilon.$$

A contradiction follows from the arbitrariness of ε. Thus $F^*(E) \leq F^{e*}(E)$, and as the reverse inequality is obvious, the proposition is proved.

EXAMPLE 4.8. Let C be the Cantor ternary set in $A = [0, 1]$, and let F be the Cantor function on A (see [6, Example 5.3.11]) viewed as an additive function on the family of all subcells of A. For each $x \in A$, let $\omega(x)$ be the distance from x to C. As C is a closed negligible set, the function $\omega : x \longmapsto \omega(x)$ is an e-gage on A. It follows that $F^{e*}(A) = 0$.

On the other hand, the Cantor function on A is the canonical example of an increasing continuous function which is not absolutely continuous. Thus, by [6, Propositions 6.4.6 and 6.4.5], the function F is not AC_*. In particular, $F^*(A) > 0$. An application of Proposition 4.9 below actually shows that $F^*(A) = F^*(C) = 1$.

PROPOSITION 4.9. *Let F be a continuous additive function on the family of all subcells of a cell $A \subset \mathbf{R}^m$. Then $F^*(A)$ is the usual variation of F on A.*

PROOF. If V_F denotes the usual variation of F, then $F^{\alpha,\delta}(A) \leq V_F(A)$ for each $\alpha > 0$ and each gage δ on A. We conclude $F^*(A) \leq V_F(A)$, and proceeding towards a contradiction, assume $F^*(A) < V_F(A)$. There are nonoverlapping cells $D_1, \ldots, D_n$ such that $\bigcup_{k=1}^n D_k = A$ and $F^*(A) < \sum_{k=1}^n |F(D_k)|$. Choose a positive $\alpha < 1$ and find a gage δ on A so that

$$F^{\alpha,\delta}(A) < \sum_{k=1}^n |F(D_k)|.$$

Given $\varepsilon > 0$, it follows from [6, Lemma 11.3.4 and Proposition 11.3.7] that in each D_k there is an α-shapely δ-fine partition $P_k = \{(A_1^k, x_1^k), \dots, (A_{p_k}^k, x_{p_k}^k)\}$ such that

$$\sum_{i=1}^{p_k} |F(A_i^k)| \geq \left|F\Big(\bigcup_{i=1}^{p_k} A_i^k\Big)\right| > |F(D_k)| - \frac{\varepsilon}{n}.$$

Clearly $\bigcup_{k=1}^{n} P_k$ is an α-shapely δ-fine partition in A, and so

$$F^{\alpha,\delta}(A) \geq \sum_{k=1}^{n} \sum_{i=1}^{p_k} |F(A_i^k)| > \sum_{k=1}^{n} |F(D_k)| - \varepsilon.$$

Since ε is arbitrary, this is a contradiction.

In view of [6, Propositions 6.4.6 and 6.4.5], the next corollary follows immediately from Proposition 4.9. Together with Theorem 4.4, it illustrates the difference between the gage and Lebesgue integrals.

COROLLARY 4.10. *Let F be a continuous additive function on the family of all subcells of a cell $A \subset \mathbf{R}^m$. The following conditions are equivalent:*

1. *F is AC_* and $F^*(A) < +\infty$;*
2. *F is AC_* and $F^{e*}(A) < +\infty$;*
3. *F' exists almost everywhere in A, and F is its indefinite Lebesgue integral.*

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