A Strong Kind of Riemann Integrability

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In 1943, Herbert Robbins [**2**] pointed out that the Riemann sums

$$\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$$

that are used to approximate an integral may continue to do so if one drops the requirement that the sequence $x_0, x_1, \ldots, x_n$ is increasing. There seems to have been little attention paid to Robbins's observation over the years, but it is worthwhile revisiting it nonetheless.

This is easy to see in some cases. Suppose f is continuous on an interval $[c, d]$ with F as its indefinite integral. Take a subinterval $[a, b] \subset [c, d]$ and any sequence of points in $[c, d]$

$$a = x_0, x_1, x_2, \ldots, x_n = b.$$

We can use the mean-value theorem to choose associated points ξ_i^* between x_{i-1} and x_i so that

$$F(x_i) - F(x_{i-1}) = f(\xi_i^*)(x_i - x_{i-1}).$$

Then, evidently,

$$\int_a^b f(x)\,dx = F(b) - F(a) = \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] = \sum_{i=1}^{n} f(\xi_i^*)(x_i - x_{i-1}).$$

It matters little here how the points $x_0, x_1, x_2, \ldots, x_n$ are arranged provided the sequence starts at a and ends at b.

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If one wishes to choose arbitrary associated points ξ_i between x_{i-1} and x_i, then this will introduce an error. The way of controlling the error is to restrict the choice of sequence by placing an upper bound on the variation

$$\sum_{i=1}^{n} |x_i - x_{i-1}|$$

of the sequence and then requiring the points to be sufficiently close together. Since not all Riemann integrable functions allow this generalization, this leads to a stricter notion of integrability which the following definition captures.

Definition. A real-valued function f is *super-Riemann integrable* on an interval $[a, b]$ provided that there is a number I so that, for every $\epsilon > 0$ and $C > 0$, there is a $\delta > 0$ with the property that

$$\left| I - \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) \right| < \epsilon$$

for any choice of points $x_0, x_1, \ldots, x_n$ and $\xi_1, \xi_2, \ldots, \xi_n$ from $[a, b]$ satisfying

$$\sum_{i=1}^{n} |x_i - x_{i-1}| \leq C,$$

where $a = x_0$, $b = x_n$, $0 < |x_i - x_{i-1}| < \delta$ and each ξ_i belongs to the interval with endpoints x_i and x_{i-1} for $i = 1, 2, \ldots, n$.

Robbins's theorem

Robbins proved that all continuous functions are super-Riemann integrable. Like many young men at that dramatic period in history, even recent Harvard Ph.D.s, Robbins was busy in other pursuits: he had joined the war effort by enlisting in the US Naval Reserve (not as a mathematician). After the war he began a long and distinguished career as a mathematical statistician and likely never had to teach college level calculus again. It seems it never occurred to him to characterize completely the class of functions that possess this strong integrability condition. We can do it for him, but it seems only fair to give him credit for the full theorem, both directions.

Robbins's Theorem. *A function f is super-Riemann integrable on an interval $[a, b]$ if and only if f is continuous there.*

Proof. Suppose first that f is continuous. Then f is Riemann integrable in the conventional sense. We prove (using a different method than that chosen by Robbins) that f also satisfies the super-integrability condition. Let $\epsilon > 0$ and $C > 0$ be given. Take δ sufficiently small that $|f(x) - f(y)| < \epsilon/C$, if x and y are points of $[a, b]$ for which $|x - y| < \delta$.

Write $F(x) = \int_a^x f(t)\,dt$. Suppose that $a \leq x \leq \xi \leq y \leq b$ and $0 < y - x < \delta$. Then, by the mean-value theorem, there is a point ξ^* between x and y for which

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$$F(y) - F(x) = f(\xi^*)(y - x).$$

Thus we also have

$$|F(y) - F(x) - f(\xi)(y - x)| = |[f(\xi^*) - f(\xi)](y - x)| < \frac{\epsilon}{C}(y - x).$$

Then, for any choice of points $x_0, x_1, \ldots, x_n$ and $\xi_1, \xi_2, \ldots, \xi_n$ from $[a, b]$ with the properties in the statement of the definition,

$$\begin{aligned}
&\left|\int_a^b f(x)\,dx - \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1})\right| \\
&\quad = \left|F(b) - F(a) - \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1})\right| \\
&\quad = \left|\sum_{i=1}^n [F(x_i) - F(x_{i-1}) - f(\xi_i)(x_i - x_{i-1})]\right| \\
&\quad \le \sum_{i=1}^n |F(x_i) - F(x_{i-1}) - f(\xi_i)(x_i - x_{i-1})| \\
&\quad < \frac{\epsilon}{C}\sum_{i=1}^n |x_i - x_{i-1}| \le \epsilon.
\end{aligned}$$

That completes the proof in one direction.

In the other direction let us suppose that f is super-Riemann integrable on $[a, b]$ and, contrary to what we want to prove, that there is a point z of discontinuity of f in the interval. We assume that $a < z < b$ and derive a contradiction. (The cases $z = a$ and $z = b$ are handled similarly.) Then there must be a positive number $\eta > 0$ so that, given any points z_1 and z_2, with $z_1 < z < z_2$, the interval $[z_1, z_2]$ contains points c_1 and c_2 for which $|f(c_1) - f(c_2)| > \eta$. Our strategy is to construct a Riemann sum that visits two suitably chosen points z_1 and z_2 repeatedly.

We apply the super-integrability hypothesis using $I = \int_a^b f(x)\,dx$, $\epsilon = \eta/4$, and $C = b - a + 4$ obtaining a δ, with $0 < \delta < 1$, that meets the conditions of the definition on $[a, b]$. Choose points $z_1 < z < z_2$ so that $z_2 - z_1 < \delta$, and then select c_1 and c_2 in the interval $[z_1, z_2]$ for which $f(c_1) - f(c_2) > \eta$. Construct a sequence

$$a = x_0 < x_1 < \cdots < x_p = z_1,$$

along with associated points $\{\xi_i\}$ so that $0 < x_i - x_{i-1} < \delta$ and so that

$$\left|\int_a^{z_1} f(x)\,dx - \sum_{i=1}^p f(\xi_i)(x_i - x_{i-1})\right| < \eta/4.$$

This actually just uses the Riemann integrability of the function f on the interval $[a, z_1]$.

Choose the least integer r so that

$$r(z_2 - z_1) > 1.$$

Note that

$$1 < r(z_2 - z_1) = (r - 1)(z_2 - z_1) + (z_2 - z_1) \le 1 + (z_2 - z_1) < 1 + \delta < 2.$$

Using r, continue the sequence $\{x_i\}$ by defining points

$$x_p = x_{p+2} = x_{p+4} = \cdots = x_{p+2r} = z_1,$$

and

$$x_{p+1} = x_{p+3} = x_{p+5} = \cdots = x_{p+2r-1} = z_2.$$

Write $\xi_{p+2j} = c_2$ and $\xi_{p+2j-1} = c_1$ for $j = 1, 2, \dots, r$. Finally, complete the sequence $\{x_i\}$ by selecting points

$$z_1 = x_{p+2r} < x_{p+2r+1} < \cdots < x_{n-1} < x_n = b$$

along with associated points $\{\xi_i\}$ so that

$$\left| \int_{z_1}^{b} f(x)\,dx - \sum_{i=p+2r+1}^{n} f(\xi_i)(x_i - x_{i-1}) \right| < \eta/4.$$

Consider now the sum

$$\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}),$$

taken over the entire sequence thus constructed. Observe that

$$\begin{aligned}
\sum_{i=1}^{n} |x_i - x_{i-1}| &= \sum_{i=1}^{p}(x_i - x_{i-1}) + \sum_{i=p+1}^{p+2r} |x_i - x_{i-1}| + \sum_{i=p+2r+1}^{n} (x_i - x_{i-1}) \\
&= (z_1 - a) + 2r(z_2 - z_1) + (b - z_1) \\
&= (b - a) + 2r(z_2 - z_1) \le (b - a) + 4 = C.
\end{aligned}$$

Thus the points chosen satisfy the conditions of the definition for the δ selected and we must have

$$\left| \int_{a}^{b} f(x)\,dx - \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) \right| < \epsilon < \eta/4.$$

On the other hand

$$\begin{aligned}
&\left[\int_a^{z_1} f(x)\,dx + \int_{z_1}^{b} f(x)\,dx - \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) \right] \\
&\quad = \left[\int_a^{z_1} f(x)\,dx - \sum_{i=1}^{p} f(x_i)(x_i - x_{i-1}) \right] \\
&\qquad + \left[\int_{z_1}^{b} f(x)\,dx - \sum_{i=p+2r+1}^{n} f(\xi_i)(x_i - x_{i-1}) \right] \\
&\qquad - \left[\sum_{i=p+1}^{p+2r} f(\xi_i)(x_i - x_{i-1}) \right].
\end{aligned}$$

From this we deduce that

$$\left|\sum_{i=p+1}^{p+2r} f(\xi_i)(x_i - x_{i-1})\right| < 3\eta/4.$$

But a direct computation of this sum shows that

$$\sum_{i=p+1}^{p+2r} f(\xi_i)(x_i - x_{i-1}) = [f(c_1) - f(c_2)]r(z_2 - z_1) > \eta r(z_2 - z_1) > \eta.$$

This contradiction completes the proof. ■

An application to change of variables

Robbins mentions in [**2**] that the idea for super-Riemann integrabilty arose from the change of variables formula. He did not supply an application, but we can. It seems most likely that some or all of this well-known theorem is what he had in mind.

Change of Variables Theorem. *Let g be Riemann integrable on an interval $[a, b]$, let G be its indefinite integral, and suppose that f is continuous on $G([a, b])$. Then*

$$\int_{G(a)}^{G(b)} f(x)\,dx = \int_a^b f(G(t))dG(t) = \int_a^b f(G(t))g(t)\,dt, \tag{1}$$

where the first and third integrals exist in the Riemann sense and the second in the Riemann-Stieltjes sense.

Proof. We use the super-Riemann integrability of f on $G([a, b])$ to show that the second integral in formula (1) exists as a Riemann-Stieltjes integral and equals the first.

Let $\epsilon > 0$ and choose a number M_1 large enough so that $|g(t)| < M_1$ for all $t \in [a, b]$. Take $C = M_1(b - a)$. Choose $\delta_1 > 0$ so that

$$\left|\int_A^B f(t)\,dt - \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})\right| < \epsilon.$$

for any choice of points $x_0, x_1, \ldots, x_n$ and $\xi_1, \xi_2, \ldots, \xi_n$ from $G([a, b])$ satisfying

$$\sum_{i=1}^{n} |x_i - x_{i-1}| \le C,$$

where $A = x_0$, $B = x_n$, $0 < |x_i - x_{i-1}| < \delta_1$ and each ξ_i belongs to the interval with endpoints x_i and x_{i-1} for $i = 1, 2, \ldots, n$. (This is a slight variant on the definition of super-integrability. The proof of Robbins's Theorem shows that this variant is also true for continuous functions.)

Since G is continuous we may choose $\delta_2 > 0$ so that if s and t are points of $[a, b]$ for which $|s - t| < \delta_2$ then necessarily $|G(s) - G(t)| < \delta_1$. Choose any points $a =$

$t_0 < t_1 < \cdots < t_n = b$ and $t_{i-1} \leq \tau_i \leq t_i$ for which $0 < t_i - t_{i-1} < \delta_2$ and consider the Riemann-Stieltjes sum

$$\sum_{i=1}^{n} f(G(\tau_i))[G(t_i) - G(t_{i-1})].$$

We can adjust this sum (without changing its value) so that the associated point τ_i in the interval $[t_{i-1}, t_i]$ is always at an endpoint. Should this not be the case just note that the identity

$$f(G(\tau_i))[G(t_i) - G(t_{i-1})] = f(G(\tau_i))[G(\tau_i) - G(t_{i-1})] + f(G(\tau_i))[G(t_i) - G(\tau_i)]$$

allows us to rewrite the sum with this endpoint property.

Let $x_i = G(t_i)$, $\xi_i = G(\tau_i)$. Note that $x_0 = G(a)$, $x_n = G(b)$, and that

$$|x_i - x_{i-1}| = |G(t_i) - G(t_{i-1})| < \delta_1.$$

We also have

$$\sum_{i=1}^{n} |x_i - x_{i-1}| = \sum_{i=1}^{n} |G(t_i) - G(t_{i-1})| \leq M_1 \sum_{i=1}^{n} |t_i - t_{i-1}| = M_1(b-a) = C.$$

Consequently, by our choice of δ_1, we have

$$\left| \int_{G(a)}^{G(b)} f(t)\,dt - \sum_{i=1}^{n} f(G(\tau_i))[G(t_i) - G(t_{i-1})] \right|$$
$$= \left| \int_{G(a)}^{G(b)} f(t)\,dt - \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) \right| < \epsilon.$$

This proves the existence of the Riemann-Stieltjes integral and establishes the first half of formula (1).

We now show that the third integral in formula (1) exists as a Riemann integral and equals the first. The function f is bounded on $G([a, b])$ so we can choose a number M_2 large enough so that $|f(G(t))| < M_2$ for all $t \in [a, b]$. Let $\epsilon > 0$ and choose $0 < \delta_3 < \delta_2$ so that for any points $a = t_0 < t_1 < \cdots < t_n = b$ and $t_{i-1} \leq \tau_i \leq t_i$ for which $0 < t_i - t_{i-1} < \delta_3$ we must have

$$\sum_{i=1}^{n} |G(t_i) - G(t_{i-1}) - g(\tau_i)(x_i - x_{i-1})| < \epsilon/M_2.$$

This follows from the fact that G is an indefinite Riemann integral of g.

Putting these together we can now compare the Riemann sums for the integral

$$\int_a^b f(G(t))g(t)\,dt$$

with the value of the integral

$$\int_{G(a)}^{G(b)} f(x)\,dx$$

by computing

$$\begin{aligned}
&\left|\int_{G(a)}^{G(b)} f(t)\,dt - \sum_{i=1}^{n} f(G(\tau_i))g(\tau_i)[x_i - x_{i-1}]\right| \\
&\quad \le \left|\int_{G(a)}^{G(b)} f(t)\,dt - \sum_{i=1}^{n} f(G(\tau_i))\,[G(t_i) - G(t_{i-1})]\right| \\
&\qquad + \left|\sum_{i=1}^{n} f(G(\tau_i))\,[G(t_i) - G(t_{i-1})] - \sum_{i=1}^{n} f(G(\tau_i))g(\tau_i)[x_i - x_{i-1}]\right| \\
&\quad \le \epsilon + \sum_{i=1}^{n} |f(G(\tau_i))\,[G(t_i) - G(t_{i-1})] - f(G(\tau_i))g(\tau_i)[x_i - x_{i-1}]| \\
&\quad \le \epsilon + M_2 \sum_{i=1}^{n} |G(t_i) - G(t_{i-1}) - g(\tau_i)[x_i - x_{i-1}]| < 2\epsilon.
\end{aligned}$$

This establishes that the third integral in formula (1) exists in the Riemann sense and is equal to the first. ■

This version of the change of variables formula is sufficiently general for most elementary purposes and has the advantage that it can be proved quite easily given the super integrability of continuous functions. The proofs are just manipulations of Riemann sums and do not use any elements of the theory. In particular we do not need to appeal to some other theorem for the existence of one of the integrals, nor do we use any properties of integrals apart from the Riemann sum definition.

The reader should be aware, however, that formula (1) is valid under much weaker assumptions. It is enough either that f is Riemann integrable on $G([a, b])$, or else that the Riemann-Stieltjes integral in (1) exists, or else that $(f \circ G)g$ is Riemann integrable on $[a, b]$. The proofs are rather more technical. See [**4**] for discussion, proofs, and references to the literature.

Characterizing derivatives

The application to change of variables is likely the only significant use of super-integrability that we can make, which seems to reduce the concept to that of an interesting curiosity. But there is, nonetheless, something else we can do: generalize Robbins's Theorem. While super-integrability, as we have seen, characterizes continuous functions, a slight modification characterizes derivatives.

We have, certainly, a great many characterizations of continuity. That continuous functions are precisely super-integrable is therefore of only moderate interest. But there are few characterizations of derivatives. The first explicit announcement of this as a research problem was by William Henry Young [**5**] in 1911. (See [**1**] for an account of the problem and for Young's enunciation of it.) Our theorem characterizes derivatives in terms of a version of super-integrability.

Characterization of Derivatives Theorem. *A real-valued function f on an interval $[a, b]$ is an exact derivative if and only if it satisfies the following strong integrability property: there is a number I so that, for every $\epsilon > 0$ and $C > 0$, there is a* *positive function δ on $[a, b]$ with the property that*

$$\left| I - \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) \right| < \epsilon, \tag{2}$$

for any choice of points $x_0, x_1, \ldots, x_n$ and $\xi_1, \xi_2, \ldots, \xi_n$ from $[a, b]$ satisfying

$$\sum_{i=1}^{n} |x_i - x_{i-1}| \leq C,$$

where $a = x_0$, $b = x_n$, $0 < |x_i - x_{i-1}| < \delta(\xi_i)$ and each ξ_i belongs to the interval with endpoints x_i and x_{i-1} for $i = 1, 2, \ldots, n$.

In one direction (as is often the case with generalizations) the proof is easier than it was for Robbins's theorem. If $F'(x) = f(x)$ at each point x of $[a, b]$, select $\delta(x) > 0$ so that

$$\left| \frac{F(z) - F(y)}{z - y} - f(x) \right| < \frac{\epsilon}{C}$$

whenever $x - \delta(x) < y < z < x + \delta(x)$. Then, with $I = F(b) - F(a)$, the inequality (2) is easy to check.

In the other direction, if f is integrable in this strong (and perhaps strange) sense, then it has an indefinite integral F which serves as an antiderivative for f. If there is a point z where $F'(z) = f(z)$ fails, we construct Riemann sums that loop repeatedly about that point (similar to the situation for the discontinuity point z in Robbins's theorem). Thus essentially the same method of proof works here too.

This theorem can stand alone as a characterization of derivatives, but is more profitably interpreted within the context of the Henstock-Kurzweil integral. (For an account of this integration theory see [**3**].) Readers familiar with that integral will spot immediately the connection. This adds another characterization of derivatives to a currently small collection and provides a new motivation for the Henstock-Kurzweil integral itself.

We can also apply this theorem to a version of the Change of Variables Theorem. Rewrite formula (1) as

$$F(G(b)) - F(G(a)) = \int_a^b F'(G(t))g(t)\,dt, \tag{3}$$

a form familiar to calculus students. We have established this identity for g Riemann integrable and for F continuously differentiable. By the same methods, we can prove that it holds for all differentiable functions F and all g that are Henstock-Kurzweil integrable. Since an exact derivative F' need not be Riemann integrable, nor even Lebesgue integrable, this is a considerable extension obtained by elementary methods. Details and references for all of the material of this section are in [**4**].

Summary. The usual definition of the Riemann integral as a limit of Riemann sums can be strengthened to demand more of the function to be integrated. This super-Riemann integrability has interesting properties and provides an easy proof of a simple change of variables formula and a novel characterization of derivatives. This theory offers teachers and students of elementary integration theory a curious and illuminating detour from the usual Rieman integral.

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Proof Without Words: Partial Sums of an Arithmetic Sequence

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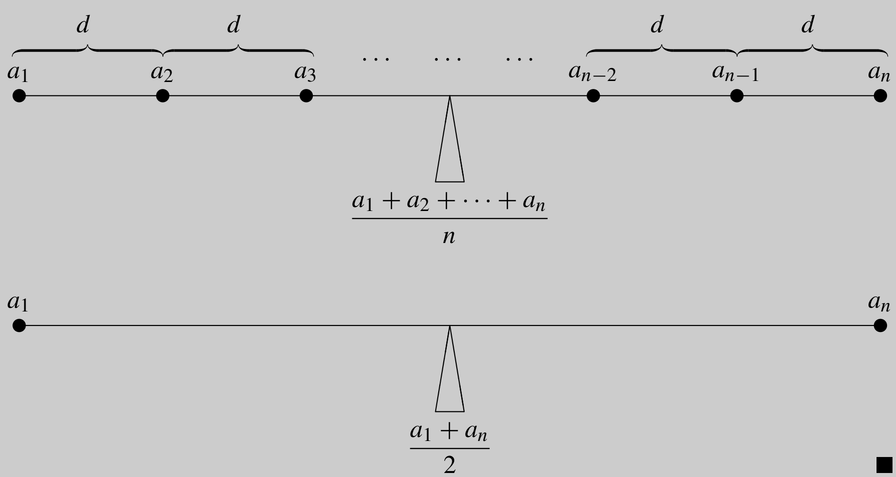
Let $a_1, a_2, a_3, \ldots$ be an arithmetic sequence. Then

$$a_1 + a_2 + \cdots + a_n = \frac{n(a_1 + a_n)}{2}$$

Proof.



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Summary. A visual proof that a partial sum of an arithmetic sequence equals the number of terms times the average of the first and last term.

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