D.R.I.P. HISTORY

www.classicalrealanalysis.com
by Alan Smithee

The genuine history of ideas is not for amateurs: it requires skill, dedication, integrity, and certainly training in the discipline of the academic historian. There is another kind of history, wherein the details are mustered in order to tell a story. One decides in advance on the narrative one wishes to tell and selects the historical events and their interpretation to further that narrative. This history is in the latter tradition. There are, naturally, far more historians of this type than of the genuine type.

The search for what should best be called *the natural integral on the real line* becomes rather heated in the late nineteenth century. By then the limitations of the popular Riemann integral became more fully appreciated. Most attention had been paid on how best to integrate unbounded functions. The discovery that there were bounded derivatives that failed to be Riemann integral meant to most that there was something seriously and fundamentally wrong with that theory.

Lebesgue's solution to this problem at the beginning of the twentieth century was to base a theory of integration on measure theory. There was enough of that to hand. Peano, Jordan, and Cantor had developed the theory of sets and their measure to a useful level. That was taken to a brilliant further step by Borel. For Lebesgue the path was clear and he developed it in a masterful way. He claimed as his motivation the fact that there existed differentiable functions F for which F' was not integrable in any of the known methods.

We know now that the premise itself is doomed to failure: measure theory alone cannot capture the integral of all functions that arise in analysis. In particular it cannot integrate all derivatives. Suppose that a function f on an interval is given and that the sets

$$\{x \in [a,b] : r \le f(x) \le s\}$$

together with their Lebesgue measures are given for all real r, s. Then the integral of f can be constructed from this information only for Lebesgue integrable functions. It can be done for all bounded derivatives; it cannot be done for *all* derivatives.

Fortunately for posterity Lebesgue did not know this in advance. His solution of the problem led to a development within a few decades of all of the tools of measure theory. His launch of this program was a greater contribution than finding the natural integral on the real line would have been.

Most teachers of the theory remark that Lebesgue abandoned the Riemann sums approach in favor of the measure-theoretic methods and that this is the source of the success of his project. In fact Lebesgue felt rather naturally compelled to show that his integral could be expressed as a limit of Riemann sums. He didn't, however, find the right filter to express this. If he had found it and thereby discovered the natural integral on the real line it might have delayed the far more important development of measure theory. But it is wrong to suggest that the success of the Lebesgue theory rests alone in the shift from Riemann sums to measures. Other techniques we now know would have generalized both integrals.

The correct and natural integral on the real line was discovered by Perron and Denjoy in the decade after Lebesgue's first work. Denjoy's idea was to extend the Lebesgue integral by a transfinite series of extensions. The series of extensions are motivated by the properties that derivatives possess. Perron's idea was to describe an integral that would have to include the Lebesgue integral and also integrate all derivatives.

Not much attention was paid to the theory of Denjoy and Perron except by some of the best analysts of the pre WWII period. The seminal book of the period,

The theory of the integral by Stanislaw Saks, New York, 1937

includes an account of the measure theory as well as the methods of Perron and Denjoy. Formidable mathematicians of the era (e.g., Banach, Steinhaus, Zygmund) were thoroughly grounded, not only in measure theory, but in these now mostly dead subjects. Later generation textbooks drop the subject. By the appearance of

Measure Theory by Paul Halmos, Van Nostrand, New York, 1950

(a standard text for many mid-twentieth century students) the subject is considered arcane and little discussed outside of certain circles[1].

In the late 1950s the simple definition of the natural integral was discovered. Henstock was working on nonabsolute integration and investigating ideas of Ward. Those ideas gelled into his formulation of a generalized Riemann integral, that he saw quickly was equivalent to the Denjoy-Perron integral. Kurzweil, at the same time, was investigating differential equations on the real line and observed the same simple formal definition of an integral that would express what he needed.

So by 1960 or so the stage was set for the introduction into the undergraduate curriculum of the natural integral on the real line. The definition was simple and natural enough for introduction at an early level. The connection with the fundamental theorem of the calculus was simpler and more compelling than in either the Riemann or Lebesgue integration theories. Measure theory could be introduced early on in simple ways or deferred until later. The students would be spared the confused mess caused by having numerous integration theories in place of one unifying theory: the Riemann integral, the improper Riemann

[1] A certain critic of the time preferred to refer to these "circles" by the more arrogant term "*cul de sac.*"

integral, the Lebesgue integral, the Denjoy-Perron integral with their oddly overlapping domains would disappear.

So why didn't it happen? There were advocates.

> Henstock, Ralph "A Riemann-type integral of Lebesgue power." Canad. J. Math. 20 1968 79–87.

offered a very readable and lucid treatment that might have had an influence if the curricula of the time had been less inflexible. At about this time, according to reliable reports, Henstock announced at an international meeting that "the Lebesgue integral is dead." It is curious that this statement was met largely with instant derision rather than any attempt to understand. Henstock meant by no means that measure-theoretic methods should be dropped; indeed he used them himself extensively. He meant that as a special integral on the real line, the Lebesgue integral should be replaced. Indeed the integral would disappear as a definition and reappear as a theorem: for all absolutely integrable functions f the integral

$$\int_a^b f(x)\,dx$$

can be constructed from the sets

$$\{x \in [a,b] : r \leq f(x) \leq s\}$$

together with their Lebesgue measures for all real r, s. Later, in graduate school, the student could learn that this theorem serves as a useful definition of an integral in situations where no other structure than a measure space is available.

At about the same time appeared the memoir

> McShane, E. J. *A Riemann-type integral that includes Lebesgue-Stieltjes, Bochner and stochastic integrals.* Memoirs of the American Mathematical Society, No. 88 American Mathematical Society, Providence, R.I. 1969 54 pp.

which suggested just how flexible the methods were but attracted attention mostly from specialists. Much later

> Bartle, Robert G., "Return to the Riemann integral." Amer. Math. Monthly 103 (1996), no. 8, 625–632.

seemed at its appearance to suggest that there was some mainstream interest in the natural integral. But it is hard to detect any changes in the usual textbooks and the usual teaching of the subject.

The first formal attempt at influencing the undergraduate curriculum might be considered the beginning of the D.R.I.P. program. Robert Bartle, Ralph Henstock, Jaroslav Kurzweil, Eric Schechter, Stefan Schwabik, and Rudolf Výborný distributed a letter to publishers' representatives at the Joint Mathematics Meetings in San Diego, California, in January 1997. Their letter is available online at this website:

http://www.math.vanderbilt.edu/ schectex/ccc/gauge/letter/

More recently the web site

http://www.classicalrealanalysis.com

has been constructed and contains some information on the program without itself being an advocate.