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σ-FINITE BOREL MEASURES ON THE REAL LINE*

Abstract

A characterization is given of those Borel measures on the real line that can be expressed as the total variation measure of an ACG_* function.

Let μ be a measure defined on the Borel subsets of an interval $[a,b]$. If μ is absolutely continuous with respect to Lebesgue measure (that is, if $\mu(N) = 0$ for every Borel set N of Lebesgue measure zero) and if $\mu([a,b]) < \infty$ then μ can be represented in the form

$$\mu(B) = \mu_f(B) = \int_B f'(x)\,dx \qquad (B \subset [a,b]), \tag{1}$$

where f is absolutely continuous on $[a,b]$ and μ_f is the corresponding Lebesgue-Stieltjes measure. Beginning students of analysis learn this material routinely.

It seems, though, that there has been little discussion of the σ-finite case. If μ is not finite, but is σ-finite, is there a representation similar to this available?

Part of such a representation is immediately available from the Radon-Nikodym theorem and a theorem of Lusin. Any absolutely continuous, σ-finite measure μ on $[a,b]$ can be represented as

$$\mu(B) = \int_B g(x)\,dx$$

for some measurable, finite a.e. function g. But Lusin's theorem (eg., see [1, p. 113]) asserts the existence of a continuous function f with $f' = g$ a.e. This gives

$$\mu(B) = \int_B f'(x)\,dx.$$

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This cannot, however, be considered a satisfactory generalization of the relation (1) since there is really no connection between the measure μ and its "associated" function f.

We propose to characterize those Borel measures μ which can be represented in the form

$$\mu(B) = \mu_f(B) = \int_B |f'(x)|\,dx \qquad (B \subset [a,b]),$$

where f is ACG$_*$ (i.e., generalized absolutely continuous in the sense of Saks [4, p. 231]) on $[a,b]$ and μ_f is the total variation measure associated with f. Recall that, for such a function f, the derivative f' would be Lebesgue integrable on a dense set of subintervals. Thus such measures are special. Indeed "most" measures are unlike this: in the space of measurable, a.e. finite functions, given an appropriate metric (see [2, p. 377]), the typical function is not Lebesgue integrable on any subinterval.

Let us begin by defining the total variation measure associated with any continuous function f on $[a,b]$. Let $E \subset [a,b]$, let δ be a gauge on E (i.e., δ is a positive function defined on E) and write

$$V(f,E,\delta) = \sup\left\{\sum |f(b_i) - f(a_i)|\right\},$$

where the supremum is taken over all disjoint collections $\{(a_i,b_i)\}$ of open subintervals of (a,b) for which there is a point $\xi_i \in E \cap (a_i,b_i)$ satisfying $b_i - a_i < \delta(\xi_i)$. Then write

$$\mu_f^*(E) = \inf\{V(f,E,\delta) : \ \delta \text{ is a gauge on } E\}.$$

It can be verified that μ_f^* is a metric outer measure on $[a,b]$. Since it is a metric outer measure its restriction to the Borel sets is a measure μ_f; we call μ_f the *total variation measure* associated with f.

If f is continuous and monotonic then μ_f is precisely the Lebesgue-Stieltjes measure generated by f. If f is continuous and has bounded variation then μ_f is the Lebesgue-Stieltjes measure associated with the total variation function for f. (Accounts of metric outer measures can be found in numerous texts, for example in [2] where also this method of construction, called Method III, is discussed.)

Our theorem characterizes those Borel measures on $[a,b]$ which arise in this way from a function that is ACG$_*$.

THEOREM *Let f be ACG$_*$ on an interval $[a,b]$. Then the total variation measure $\mu = \mu_f$ associated with f has the following properties:*

a. μ is a σ-finite Borel measure on $[a,b]$.

b. μ is absolutely continuous with respect to Lebesgue measure.

c. There is a sequence of closed sets F_n whose union is all of $[a,b]$ such that $\mu(F_n) < \infty$ for each n.

d. $\mu(B) = \mu_f(B) = \int_B |f'(x)|\,dx$ for every Borel set $B \subset [a,b]$.

Conversely, if a measure μ satisfies conditions (a)–(c) then there exists an ACG_* function f for which the representation (d) is valid.

Proof. Suppose first that f is ACG$_*$ on the interval $[a,b]$ and let μ_f denote its total variation measure. We know already that this is a Borel measure and to verify assertion (a) we need to check that it is σ-finite. But this follows from (c) and so it will be enough to check that.

Fix $\epsilon > 0$. Since f is ACG$_*$ it can be represented as a Denjoy-Perron integral

$$f(x) - f(a) = \int_a^x g(x)\,dx$$

where $g = f'$ a.e. By the well-known Saks–Henstock Lemma (e.g., see [3, p. xxx]) for any $\epsilon > 0$ there is a gauge δ on $[a,b]$ with the property that

$$\sum_{i=i}^{n} |f(b_i) - f(a_i) - g(\xi_i)(b_i - a_i)| < \epsilon \tag{2}$$

for any sequence of disjoint subintervals (a_i, b_i) of (a,b) and points $\xi_i \in (a_i, b_i)$ with $b_i - a_i < \delta(\xi_i)$. Let

$$E_n = \{x \in [a,b] : |g(x)| \le n \text{ and } \delta(x) > 1/n\}.$$

Then E_n is an increasing sequence of subsets of $[a,b]$, whose union is all of $[a,b]$.

We prove assertion (b). Let N be a subset of $[a,b]$ of measure zero. There must be an open set $G \supset N$ so that $|G| < \epsilon$. Choose a gauge δ' on E_n so that $\delta' \le \delta$ and so that $(x - \delta'(x), x + \delta'(x)) \subset G$ for every $x \in N$. We estimate $V(f, E_n \cap N, \delta')$. Consider any sequence of intervals $\{(a_i, b_i)\}$ for which there is a point $\xi_i \in E_n \cap N \cap (a_i, b_i)$ satisfying $b_i - a_i < \delta'(\xi_i)$. By the way δ' was defined we see that each interval (a_i, b_i) appearing is a subset of G. Thus, using (2) and the fact that each $\xi_i \in E_n$, we have

$$\sum_{i=i}^{n} |f(b_i) - f(a_i)| < \sum_{i=i}^{n} |g(\xi_i)|\,(b_i - a_i) + \epsilon \le n|G| + \epsilon < \epsilon(n+1).$$

It follows that

$$\mu_f^*(E_n \cap N) \le V(f, E_n \cap N, \delta') \le \epsilon(n+1).$$

Since ϵ is arbitrary, each $\mu_f^*(E_n \cap N) = 0$. As the sequence of sets $\{E_n\}$ cover all of N it follows that $\mu_f^*(N) = 0$. This shows that μ is absolutely continuous with respect to Lebesgue measure, establishing assertion (b).

Let us prove assertion (c) by showing that, for each n, $\mu(\overline{E_n}) < \infty$. Define a gauge δ' on $\overline{E_n}$ to agree with δ on E_n and on the remaining points in $\overline{E_n}$ to be $1/n$. We estimate $V(f, \overline{E_n}, \delta')$. Consider any sequence of intervals $\{(a_i, b_i)\}$ for which there is a point $\xi_i \in \overline{E_n} \cap (a_i, b_i)$ satisfying $b_i - a_i < \delta'(\xi_i)$. By the way E_n and δ' were defined we may consider that $\xi_i \in E_n$ and that $b_i - a_i < \delta'(\xi_i)$, since we can replace any such point with a nearby one in E_n.

Thus, again using (2), we have

$$\sum_{i=i}^{n} |f(b_i) - f(a_i)| < \sum_{i=i}^{n} |g(\xi_i)|\,(b_i - a_i) + \epsilon \le n(b-a) + \epsilon.$$

It follows that

$$\mu(\overline{E_n}) \le V(f, \overline{E_n}, \delta') \le n(b-a) + \epsilon < \infty$$

as we desired. This establishes assertion (c).

Finally to prove (d) we require the representation

$$\mu_f(B) = \int_B |g(x)|\,dx. \tag{3}$$

Let E be a measurable subset of $[a, b]$, at each point x of which

$$0 \le d \le |g(x)| \le c.$$

We show that

$$\mu_f^*(E) \le c|E| \tag{4}$$

and that

$$d|E| \le \mu_f^*(E). \tag{5}$$

Choose an open set $G \supset E$ so that $|G| < |E| + \epsilon$. Choose a gauge δ' on E so that $\delta' \le \delta$ and so that $(x - \delta'(x), x + \delta'(x)) \subset G$ for every $x \in E$. We estimate $V(f, E, \delta')$. Consider any sequence of intervals $\{(a_i, b_i)\}$ for which there is a point $\xi_i \in E \cap (a_i, b_i)$ satisfying $b_i - a_i < \delta'(\xi_i)$. By the way δ' was defined we see that each interval (a_i, b_i) appearing is a subset of G. Thus, using (2), we have

$$\sum_{i=i}^{n} |f(b_i) - f(a_i)| < \sum_{i=i}^{n} |g(\xi_i)| (b_i - a_i) + \epsilon \le c|G| + \epsilon.$$

It follows that

$$\mu_f^*(E) \le V(f, E, \delta') \le c(|E| + \epsilon) + \epsilon.$$

Since ϵ is arbitrary, assertion (4) follows.

Now let δ' be *any* gauge on E. Write $\delta'' = \min\{\delta, \delta'\}$. Let $\mathcal{V}$ denote the collection of all intervals $[\alpha, \beta]$ such that (α, β) contains a point $\xi \in E$ for which $\beta - \alpha < \delta''(\xi)$. Note that $\mathcal{V}$ forms a Vitali cover of the measurable set E. By the Vitali covering theorem there must exist a disjoint sequence of intervals $\{(a_i, b_i)\}$ and points $\xi_i \in E \cap (a_i, b_i)$ with $b_i - a_i < \delta''(\xi_i)$ so that

$$\sum_i (b_i - a_i) \ge |E|.$$

Thus, using (2), we have

$$d|E| \le d \sum_i (b_i - a_i) \le \sum_i |g(\xi_i)| (b_i - a_i) \le \sum_i |f(b_i) - f(a_i)| + \epsilon.$$

It follows that

$$V(f, E, \delta') \ge V(f, E, \delta'') \ge d|E| - \epsilon$$

for every gauge δ' on E. Hence, since ϵ is arbitrary, assertion (5) follows.

¿From assertion (4) and (5) we obtain assertion (3) by ordinary measure theoretic arguments. For example if h_1 and h_2 are simple, nonnegative measurable functions on $[a, b]$ with $h_1 \le g \le h_2$ then we have

$$\int_a^b h_1(t)\, dt \le \mu_f^*(E) \le \int_a^b h_2(t)\, dt.$$

¿From this then assertion (3) follows by taking appropriate sequences of such simple functions converging to g. This completes the proof of (d) and so one direction of the theorem is established.

Let us now prove the converse of the theorem. We suppose that a measure μ is given with the properties (a), (b) and (c) and we wish to construct the function f so that (d) holds. Let E_n be an expanding sequence of closed sets whose union is equal to $[a, b]$ and with each $\mu(E_n) < \infty$.

By the Radon-Nikodym theorem there is a nonnegative, measurable function g_1 on $[a,b]$ so that

$$\mu(E_1 \cap B) = \int_B g_1(x)\,dx.$$

We can assume that $g_1 = 0$ off of E_1. Construct a function F_1, absolutely continuous on $[a,b]$, so that
(i) $F_1(a) = F_1(b)$.
(ii) $F_1'(x) = |g_1(x)|$ for a.e. $x \in (a,b)$.
(iii) $|F_1(x) - F(a)| \leq b - a$ for each $x \in (a,b)$.

To achieve (ii) it would be enough to take $F_1(x) = \int_a^x g_1(t)\,dt$. But this will not give (i) and (iii). For that simply choose an appropriate function h that assumes only the values ± 1 and write $F_1(x) = \int_a^x g_1(t)h(t)\,dt$. (For example partition $[a,b]$ into a finite number of subintervals in the correct manner and set h to be $+1$ and -1 on alternate intervals.)

Note that F_1 is constant on each interval complementary to E_1 in $[a,b]$. On these complementary intervals we now change F_1. Again, by the Radon-Nikodym theorem there is a nonnegative, measurable function g_2 on $[a,b]$ so that

$$\mu((E_2 \setminus E_1) \cap B) = \int_B g_2(x)\,dx.$$

We can assume that $g_2 = 0$ off of E_2. Construct a function F_2, absolutely continuous on $[a,b]$, that agrees with F_1 on E_1 and, for each interval (α,β) complementary to E_1, we arrange that
(i) $F_2(\alpha) = F_2(\beta) = F_1(\alpha) = F_1(\beta)$.
(ii) $F_2'(x) = |g_2(x)|$ for a.e. $x \in (\alpha,\beta)$.
(iii) $|F_2(x) - F_2(\alpha)| \leq \beta - \alpha$ for $x \in (\alpha,\beta)$.
The method of construction is identical to that used to produce F_1.

This procedure is continued inductively and so defines a function F agreeing with each F_n on each E_n. We claim that F is ACG_* on $[a,b]$, that $F'(x) = g_n(x)$ for a.e. $x \in E_n \setminus E_{n-1}$ and that assertion (4) holds.

Observe that F and F_n are identical on the closed set E_n and that the oscillations of the function F on intervals complementary to E_n in $[a,b]$ form a convergent series (because of the requirement (iii) in the construction). Consequently F is AC_* on each E_n and so ACG_* on $[a,b]$.

Let us check the derivative of F at points in E_1. We have defined F_1 in such a way that $|F_1'(x)| = g_1(x)$ for a.e. $x \in E_1$. But F and F_1 differ on the intervals complementary to E_1 and this may affect the derivative, so it is not clear that $F'(x)$ and $F_1'(x)$ must agree. Note that the oscillation of F on any interval (α,β) contiguous to E_1 cannot exceed $2(\beta - \alpha)$.

Take a point $x \in E_1$ at which $F'(x)$ exists and suppose, moreover, that x is a point of nonporosity[1] of E_1. Consider $(F(y_n) - F(x))/(y_n - x)$ for appropriate sequences y_n decreasing to x. (The argument on the left side is similar.) We need not worry if $y_n \in E_1$ since F and F_1 agree there. So let $y_n \in (\alpha_n, \beta_n)$ where these are intervals contiguous to E_1.

Our assumptions allow us to assert that

$$\frac{F(\alpha_n) - F(x)}{\alpha_n - x} \to F'(x), \tag{6}$$

$$|F(y_n) - F(\alpha_n)| < 2(\beta_n - \alpha_n) \tag{7}$$

and

$$\frac{\beta_n - \alpha_n}{\alpha_n - x} \to 0 \quad \text{and} \quad \frac{\alpha_n - x}{y_n - x} \to 1. \tag{8}$$

Putting (6)–(8) together we have, after some simple computations, that

$$\frac{F(y_n) - F(x)}{y_n - x} \to F_1'(x). \tag{9}$$

From (9) we have now that $F'(x) = F_1'(x) = |g_1(x)|$ for a.e. $x \in E_1$ since a.e. point in E_1 is both a point of existence of $F'(x)$ and a point of nonporosity of E_1.

Precisely the same argument applies to points in $E_2 \setminus E_1$ at the next stage of the construction and so on inductively. Thus $|F'(x)|$ is a.e. equal to $g_n(x)$ on the set $E_n \setminus E_{n-1}$. For any Borel set B we have

$$\mu(B \cap (E_n \setminus E_{n-1})) = \int_{B\cap(E_n \setminus E_{n-1})} g_n(t)\,dt = \int_{B\cap(E_n \setminus E_{n-1})} |F'(t)|\,dt.$$

From this the representation

$$\mu(B) = \int_B |F'(t)|\,dt$$

now follows. The final step requires merely for us to note that

$$\mu_F(B) = \int_B |F'(t)|\,dt$$

for any function F that is ACG$_*$. □

[1] Porosity is defined, for example, in [2, p. 325].

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