Quiz

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Solve each of the following questions using

- THE INTEGRAL[1].
- The Riemann integral.
- The Lebesgue integral.

#1. What is

$$\int_0^1 x^2 \, dx?$$

SOLUTION ON PAGE 2.

#2. What is

$$\int_0^1 \frac{1}{\sqrt{x}} \, dx?$$

SOLUTION ON PAGE 4.

#3. What is

$$\int_{-1}^1 \frac{1}{\sqrt{|x|}} \, dx?$$

SOLUTION ON PAGE 6.

#4. What is your point?

SOLUTION ON PAGE 7.

#5. Why is THE INTEGRAL claimed to be easier than the other two (or is it three) integrals?

SOLUTION ON PAGE 7.

[1] i.e., the natural integral on the real line, otherwise known as the Denjoy integral, the Perron integral, the Denjoy-Perron integral, the restricted integral of Denjoy, the Denjoy total, the Henstock integral, the Kurzweil integral, the Henstock-Kurzweil integral, the Kurzweil-Henstock integral, the generalized Riemann integral, the Riemann-complete integral, the gage integral, the gauge integral, etc. We call it here simply THE INTEGRAL.

1 Problem #1

Solution

$$\int_0^1 x^2\,dx?$$

- **The integral**:

  Observe that, with $f(x) = x^2$ and $F(x) = x^3/3$, we have $F'(x) = f(x)$ at every point of the interval $[0,1]$.

  Consequently we can use the following theorem of integration theory (known sometimes as the fundamental theorem of the calculus).

  > *Theorem A. If $F'(x) = f(x)$ at every point of $[a,b]$ then f is integrable on $[a,b]$ and the value of the integral is exactly*
  >
  > $$\int_a^b f(x)\,dx = F(b) - F(a).$$

  We conclude that f is integrable on [0,1] and the value of the integral is $F(1) - F(0) = 1/3$.

- **The Riemann integral:**

  Observe that, with $F(x) = x^3/3$, we have $F'(x) = f(x)$ at every point of the interval $[0,1]$. We cannot use Theorem A since that is false for the Riemann integral. But there is a rather pathetic variant we can use:

  > *Theorem A^R. If $F'(x) = f(x)$ at every point of $[a,b]$ and if f is integrable on $[a,b]$ then the value of the integral is exactly $F(b) - F(a)$.*

  Thus we check first that f is continuous on $[0,1]$, we apply a theorem asserting that continuous functions are Riemann integrable and finally we conclude that f is integrable on $[0,1]$ and the value of the integral is $F(1) - F(0) = 1/3$.

- **The Lebesgue integral:**

  Observe that, with $F(x) = x^3/3$, we have $F'(x) = f(x)$ at every point of the interval $[0,1]$. We cannot use Theorem A since that is false for both the Riemann integral and the Lebesgue integral.

  > *Theorem A^L. If $F'(x) = f(x)$ at every point of $[a,b]$ and if f is integrable on $[a,b]$ then the value of the integral is exactly $F(b) - F(a)$.*

This too is a pathetic variant. But, even so, it is very seldom proved in analysis courses and the average graduate student would be unlikely to know the statement or be able to prove it. (Most would believe it but be amazed at not being able to prove it.)

But there is a weaker variant we can use:

> *Theorem* B^L*. If* $F'(x) = f(x)$ *at [almost] every point of* $[a, b]$ *and if F is absolutely continuous on* $[a, b]$ *then* f *is integrable on* $[a, b]$ *and the value of the integral is exactly* $F(b) - F(a)$*.*

Thus we check first that F is Lipschitz on $[0, 1]$ with Lipschitz constant 2 (just use the mean-value theorem) and then invoke a theorem asserting that Lipschitz functions are absolutely continuous. Again we conclude that f is integrable on $[0, 1]$ and the value of the integral is $F(1) - F(0) = 1/3$.

2 Problem #2

Solution

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx?$$

Note: The student will likely be disturbed by the fact that the integrand is undefined at the point $x = 0$. This is a mystery. All Riemann type theories are uninfluenced by the value of the function at a single point. So for both the integral and the Riemann integral simply ignore the point or assign some other convenient value there. For the Lebesgue integral (and indeed for THE INTEGRAL) a set of measure zero may be ignored, so the single point $x = 0$ shouldn't distress graduate students.

- **The integral**:

  Observe that, with $F(x) = 2\sqrt{x}$, we have $F'(x) = f(x)$ at every point of the interval $[0, 1]$ with one exception. Consequently we cannot use Theorem A above, since that requires a derivative at *every* point. Instead we have the following theorem of integration theory, a modification of Theorem A.

  > *Theorem C. If F is a continuous function on $[a, b]$ and if $F'(x) = f(x)$ at every point of $[a, b]$ with a number of exceptions [at most countably many exceptions], then f is integrable on $[a, b]$ and the value of the integral is exactly $F(b) - F(a)$.*

  We conclude that f is integrable on $[0, 1]$ and the value of the integral is $F(1) - F(0) = 2$.

- **The Riemann integral**:

  Trick question! You cannot use the Riemann integral for unbounded functions. A properly disciplined calculus student will use the improper version of the Riemann integral.

  Observe that, with $F(x) = 2\sqrt{x}$, we have $F'(x) = f(x)$ at every point of the interval $[0, 1]$ with one exception, at $x = 0$. But this function f is unbounded on $[0, 1]$ and so f is not Riemann integrable. But most students have learned an extension of the Riemann integral, known as "the improper integral." Theorem C is false for both the Riemann and improper Riemann integrals.

  Instead the student must fall back on the following canonical ritual of the elementary calculus. This function f is continuous on $[t, 1]$ for all $t > 0$ so is Riemann integrable. By applying Theorem A^R determine that this integral has value $F(1) - F(t) = 2 - 2\sqrt{t}$. Take the limit as $t \to 0+$ from the right and, using the continuity of the function F, obtain the finite value $F(1) - F(0) = 2$.

Finally we conclude that f is "integrable" on $[0, 1]$ and the value of the integral is $F(1) - F(0) = 2$.

- **The Lebesgue integral**:

  Observe that, with $F(x) = 2\sqrt{x}$, we have $F'(x) = f(x)$ at every point of the interval $[0, 1]$ with one exception. We cannot use Theorem C since that is false for the Lebesgue integral. We cannot (surprisingly for some students) use the "improper" prescribed procedure just used either, for that is not available for the Lebesgue integral. [The ritual which was nearly a religious obligation in elementary calculus is forbidden to acolytes of the Lebesgue integral.]

  We could try for this theorem:

  > *Theorem C^L. If F is a continuous function on $[a, b]$, if $F'(x) = f(x)$ at every point of $[a, b]$ with countably many exceptions and if f is integrable on $[a, b]$, then the value of the integral is exactly $F(b) - F(a)$.*

  This is true, but it is not at all taught in typical graduate courses. So the student has little recourse but to return to Theorem A^L and verify that the function F is absolutely continuous.

  This function is not Lipschitz so that fact will need verification, likely from the definition. Again we conclude (after some considerable labor) that f is integrable on $[0, 1]$ and the value of the integral is $F(1) - F(0) = 2$.

  An alternate method would be to use the monotone convergence theorem and truncate the function f by $f_n(x) = \min\{n, f(x)\}$, then letting $n \to \infty$ and seeing that f_n increases to f. But in either the case the student is embarrassed by the considerable machinery of Lebesgue theory that needs to be brought to bear on a problem which caused much less grief as a freshman student.

3 Problem #3

Solution

$$\int_{-1}^{1} \frac{1}{\sqrt{|x|}}\, dx?$$

Note: Once again he student will likely be alarmed by the fact that the integrand is undefined at the center of the interval. Indeed most calculus texts would avoid this; they are less bothered if the undefined points occur at an endpoint.

- **The integral**:

  It is easy enough to tailor an indefinite integral: take $F(x) = 2\sqrt{x}$ for $x \geq 0$ and as $-2\sqrt{-x}$ for $x < 0$. Then F is continuous and $F'(x) = f(x)$ with one exception. Apply the usual fundamental theorem of the calculus (Theorem C) to obtain

  $$\int_{-1}^{1} \frac{1}{\sqrt{|x|}}\, dx? = F(1) - F(-1) = 4.$$

- **The Riemann integral**:

  Split this into the two integrals

  $$\int_{-1}^{1} \frac{1}{\sqrt{|x|}} = \int_{-1}^{0} \frac{1}{\sqrt{|x|}} + \int_{0}^{1} \frac{1}{\sqrt{|x|}}.$$

  Then worry. There don't seem to be any theorems about the improper Riemann integral in our calculus book that allow this, but it seems likely!

- **The Lebesgue integral**:

  See the solution for the Riemann integral. That would be ok for the Lebesgue integral.

4 Question #4

What is your point?

Well if your students have no troubles with problems #1, #2, or #3 then maybe there is no point. There are anecdotes about graduate students given problem #1 on an oral exam and freezing in a panic.

5 Question #4

Why is THE INTEGRAL claimed to be easier than the Riemann integral, the improper Riemann integral and the Lebesgue integral?

For elementary courses the answer is, it seems, merely that the integral is defined in a way that makes it closely connected with the fundamental theorem of the calculus. The definition is just formal, but it allows immediate proofs of the most useful versions of the fundamental theorem. Since these are the methods that calculus students use to compute integrals it is convenient that they can have access to these theorems.

Ultimately this formal definition inhibits any real development of the theory and the apparatus of measure theory must be introduced. Some presentations of the integral (i.e., of this natural integral on the real line) obscure this by delaying measure theory as a topic, but using measure-theoretic methods buried in the proofs.

For mathematicians, though, the greatest appeal should be in the simplicity. There is a single natural theory of integration on the real line–not a Riemann integral, an improper Riemann integral, and the Lebesgue integral, with murky relationships among the three. The natural integral serves as a good introduction to the modern methods of measure theory, since the Lebesgue measure on the real line comes in quite naturally as one of the premier tools in studying the integral. Then that theory can quite naturally be generalized for advanced purposes.

Teach THE INTEGRAL at the undergraduate level bringing in the measure theory as soon as feasible. Show how the integral (for absolutely integrable functions) can be characterized and constructed by measure-theoretic methods. Then in graduate school take the measure-theoretic methods to an abstract level. You will leave behind the nonabsolute integral where it belongs: on the real line.